SOLID GEOMETRY

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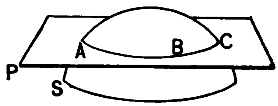
SOLID GEOMETRY.

BOOK VI. — LINES AND PLANES IN SPACE.

1. THE POSITION OF A PLANE IN SPACE. THE STRAIGHT LINE AS THE INTERSECTION OF TWO PLANES.

328. Definitions. Through three points, not in a straight line, any number of surfaces may be imagined to pass.



For example, through the points A, B, C the surfaces P and S may be imagined to pass.

329. A plane surface (also called a *plane*) is a surface which is determined by any three of its points not in a straight line.

In the figure, P represents a plane, for it is determined by the points A, B, C. But S does not represent such a surface.

A plane is, of course, supposed to be indefinite in extent.

This definition, and the following postulates, are repeated, for convenience, from the Plane Geometry.

In drawing a figure it should be remembered that a plane, like a line, has no thickness, and that it is indefinite in extent. Nevertheless, it aids the eye in understanding the figure, if we represent the plane as a rectangle, lying in perspective, and having a slight thickness.

Exercises. 522. Show that if there are given four points in space, no three being collinear, the number of distinct straight lines determined by them is six; if there are five points, the number of lines is ten.

523. Hold two pencils in such a way as to show that a plane cannot, in general, contain two straight lines taken at random in space.

330. Postulates of the Plane. (See § 29.)

1. *Three points not in a straight line determine a plane.*

2. *A straight line through two points in a plane lies wholly in the plane.*

3. *A plane may be passed through a straight line and revolved about it so as to include any assigned point in space.*

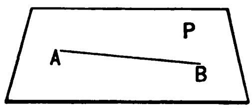
4. *A portion of a plane may be produced.*

5. *A plane is divided into two parts by any one of its straight lines, and space is divided into two parts by any plane.*

331. Solid Geometry treats of figures whose parts are not all in one plane.

PROPOSITION I.

332. Theorem. *A plane is determined by a straight line and a point not in that line.*



Given the line AB, and the point P not in that line.

To prove that AB and P determine a plane.

Proof. 1. Only one plane contains pts. A, B, and P. § 330, 1

(§ 330, 1. Three points not in a straight line determine a plane.)

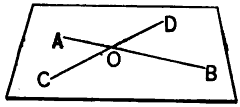
2. And that plane contains line AB. § 330, 2

(§ 330, 2. A straight line joining two points in a plane lies wholly in the plane.)

3. $\therefore$ only one plane contains AB and P.

333. Definition. Lines or points which lie in the same plane are said to be **coplanar**.

Corollaries. 1. *A plane is determined by two intersecting lines.*



Let the lines AB, CD intersect at O.

Then only one plane contains AB and C. Prop. I

And that plane contains the point O, for O lies in the line AB. § 330, 2

And since that plane contains C and O, it contains CD. § 330, 2

2. *A plane is determined by two parallel lines.*

For the parallels lie in one plane, by definition (§ 82).

And only one plane can contain these parallels, since a plane is determined by either line and any point of the other.

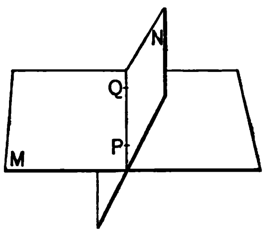
Draw the figure.

3. *If a plane contains one of two parallel lines and any point of the other, it contains both parallel lines.*

For it must be identical with the plane determined by the two parallels; otherwise more than one plane could contain either parallel and any point in the other.

Proposition II.

334. Theorem. *The intersection of two planes is a straight line.*



Given two intersecting planes, M, N.

To prove that their intersection is a straight line.

Proof. 1. Let P be a point common to M and N.

Then a pencil of lines through P, in the plane N, must lie partly on one side of M and partly on the other, because M divides space into two parts. § 330, 5

2. Hence, in general, a line connecting a point in the pencil on one side of M, with a point on the other side, must cut M at some other point than P, — say at Q.

3. Then M and N have two points in common.

4. Then every point in the straight line through P and Q lies in plane M, § 330, 2
and also in plane N, for the same reason.

5. $\therefore$ the straight line PQ is common to both planes.

6. If there were any point not in PQ, common to M and N, the planes would coincide. Prop. I

COROLLARY. *A point common to two planes lies in their line of intersection.*

Proved in step 6.

Exercises. 524. State the four methods, already mentioned, of determining a plane.

525. Is it possible for three planes to have a straight line in common? Draw a figure to illustrate.

526. If two planes have three points in common, will they necessarily coincide?

527. Four planes, no three containing the same line, intersect in pairs; how many straight lines do they determine by their intersections?

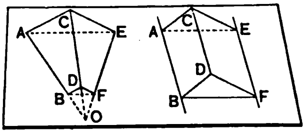
528. What is the only rectilinear polygon that is necessarily plane? Why?

529. Prove that all transversals of two parallel lines are coplanar with the parallels.

530. What is the reason that a three-legged chair is always stable on the floor while a four-legged one may not be?

Proposition III.

335. Theorem. *If three planes, not containing the same line, intersect in pairs, the three lines of intersection are either concurrent or parallel.*



Given planes *AD, CF, EB*, intersecting in *AB, CD, EF*.

To prove that *AB, CD, EF* are either concurrent or parallel.

Proof. Case I. If *CD* meets *AB*, as at *O*, to show the three lines concurrent.

1. ∵ *O* is in *AB*, it is in plane *EB*. § 330, 2
2. Similarly, ∵ *O* is in *CD*, it is in plane *CF*. Why ?
3. ∵ *O* is in planes *EB* and *CF*, *EF* passes through *O*. Prop. II, cor.
4. ∴ *AB, CD, EF* are concurrent in *O*.

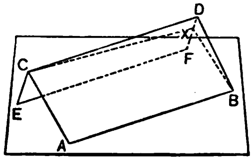
Case II. If *CD* ∥ *AB*, to show the three lines parallel.

1. If *AB* were not ∥ *EF*, *CD* would pass through their common point. Case I
2. But this is impossible, for *CD* ∥ *AB*. Given
3. If *CD* were not ∥ *EF*, *AB* would pass through their common point. Case I
4. But this is impossible, for *CD* ∥ *AB*. Given
5. ∴ as no two can meet, and as each pair is coplanar, they are parallel. Def. ∥ lines

Corollary. *If two intersecting planes pass through two parallel lines, their intersection is parallel to these lines.*

PROPOSITION IV.

336. Theorem. *Lines parallel to the same line are parallel to each other.*



Given $AB \parallel EF$, $CD \parallel EF$.

To prove that $AB \parallel CD$.

Proof. 1. AB and EF determine a plane. Prop. I, cor. 2

(A plane is determined by two $\parallel$ lines.)

2. CD and EF determine a plane. Why?

3. AB and any point C of CD determine a plane. Prop. I

4. Suppose this last plane to intersect plane ED in CX, another line than CD.

Then CX would be $\parallel$ to both EF and AB. Prop. III, cor.

(If two intersecting planes pass through two $\parallel$ lines, their intersection is $\parallel$ to these lines.)

5. But $\because CD \parallel EF$, this is impossible. Post. of parallels

(§ 85. Two intersecting straight lines cannot both be $\parallel$ to the same straight line.)

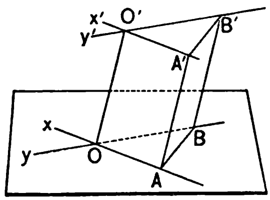
6. $\therefore CD$ is the intersection of the planes through AB and C, and EF and C,

and $\therefore CD \parallel AB$. Prop. III, cor.

Exercise. 531. Why will not the proof of this theorem as given in plane geometry apply to this case in solid geometry?

Proposition V.

337. Theorem. *If two intersecting lines are respectively parallel to two others, the angles made by the first pair are equal or supplemental to those made by the second pair.*



Given two intersecting lines x, y, respectively parallel to two other lines x', y'.

To prove that the angles made by x and y are equal or supplemental to those made by x' and y'.

Proof. 1. Suppose the intersections O and O' are joined, and from any points A, B, on x, y, parallels to OO' are drawn.

2. $\because$ OO', x, and x' are coplanar (Why ?), the parallel from A meets x' as at A'. Similarly, B' is fixed. Prop. I, cor. 3

3. Draw AB, $A'B'$.

$\because AA' \parallel OO'$, and $BB' \parallel OO'$, $\therefore AA' \parallel BB'$. Prop. IV

4. $\because OA'$, OB' are ▱s, $\therefore AA' = OO' = BB'$. I, prop. XXIV

5. $\therefore ABB'A'$ is a ▱. I, prop. XXV

6. $\therefore OA = O'A'$, $OB = O'B'$, $AB = A'B'$. I, prop. XXIV

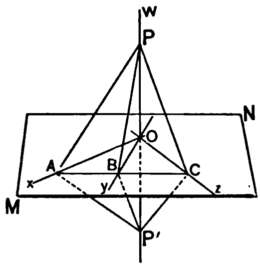
7. $\therefore \triangle ABO \cong \triangle A'B'O'$, and $\angle AOB = \angle A'O'B'$. I, prop. XII

After proving one pair of angles equal, the rest are evidently equal or supplemental by the theorems concerning vertical and supplemental angles.

2. THE RELATIVE POSITION OF A LINE AND A PLANE.

PROPOSITION VI.

338. Theorem. *If a line is perpendicular to each of two intersecting lines, it is perpendicular to every other line lying in their plane and passing through their point of intersection.*



Given x and z, two lines intersecting at O, and w perpendicular to x and to z; also y, any line through O coplanar with x, z.

To prove that $w \perp y$.

Proof. 1. On w suppose $OP' = PO$; let any transversal cut x, y, z at A, B, C; join P and P' with A, B, C.

Then $AP = AP'$, and $CP = CP'$. I, prop. XX, cor. 5

2. And $\because AC \equiv AC$,

$\therefore \triangle ACP \cong \triangle ACP'$. I, prop. XII

3. $\therefore$ by folding $\triangle ACP$ over AC as an axis, it can be brought to coincide with $\triangle ACP'$. § 57

4. $\therefore \triangle BOP \cong \triangle BOP'$, I, prop. XII

and $\angle POB$ is a rt. $\angle$, and $w \perp y$. Why?

339. Definitions. A line is said to be **perpendicular to a plane** when it is perpendicular to every line in that plane which passes through its *foot*, — *i.e.* the point where it meets the plane. The plane is then said to be **perpendicular to the line.**

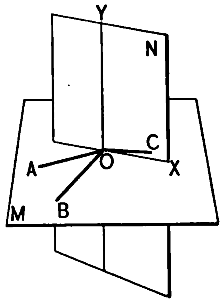
If a line meets a plane, and is not perpendicular to it, it is said to be **oblique to the plane.**

COROLLARIES. 1. *If a line is perpendicular to each of two intersecting lines, it is perpendicular to their plane.*

2. *The locus of points equidistant from two given points is the plane bisecting at right angles the line joining those points.*

PROPOSITION VII.

340. Theorem. *If a line is perpendicular to each of three concurrent lines at their point of concurrence, the three lines are coplanar.*



Given $OY \perp OA, OB, OC.$

To prove that OA, OB, OC are coplanar.

Proof. 1. Suppose M the plane determined by OA, OB; and N the plane determined by OY, OC.

Suppose that OC is not in M, and call OX the intersection of M and N.

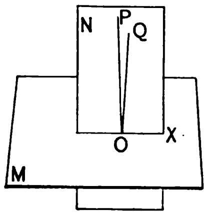
2. Then must $OY \perp OX$. Prop. VI

3. But $\because OY \perp OC$, this is impossible. Prel. prop. II

4. $\therefore$ it is absurd to suppose OC not in M with OA and OB.

COROLLARIES. 1. *Lines perpendicular to the same line at the same point are coplanar.*

2. *Through a given point in a plane there cannot be drawn more than one line perpendicular to that plane.*



Suppose OP and $OQ \perp$ plane M. Then each would be perpendicular to OX, the line of intersection of their plane N with the given plane M, thus violating prel. prop. II.

3. *Through a given point in a line there cannot be drawn more than one plane perpendicular to that line.*

For if two planes could be drawn perpendicular to the line, then three lines in each would be perpendicular to the given line, and hence the two planes would coincide.

Exercises. 532. Prove that if the hand of a clock is perpendicular to its moving axle, it describes a plane in its revolution. Prove the converse.

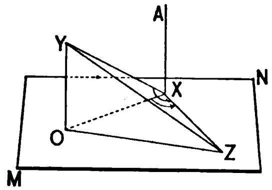
533. How many straight lines are determined by six points, three being collinear?

534. How many planes in general are determined by four points in space, no three being collinear?

535. In the left-hand figure of prop. III, suppose point O to move farther from BDF, and to continue to do so indefinitely. What is the limiting figure which the left-hand figure is approaching?

Proposition VIII.

341. Theorem. *Lines perpendicular to the same plane are parallel.*



Given OY, $XA \perp$ plane MN at O, X.

To prove that $OY \parallel XA$.

Proof. It is necessary first to show that OY, XA are coplanar; then that they are $\perp$ to OX.

1. Let $XZ \perp OX$ in plane MN, and $= OY$.

   Draw OZ, ZY, XY.

2. Then $\because XZ = OY$, $OX \equiv OX$,

   and $\angle XOY = \angle OXZ =$ rt. $\angle$,

3. $\therefore \triangle XOY \cong \triangle OXZ$,

   and $OZ = XY$. I, prop. I

4. And $\because ZY \equiv ZY$, $\therefore \triangle XYZ \cong \triangle OZY$,

   and $\angle YXZ = \angle ZOY =$ rt. $\angle$. I, prop. XII

   $\therefore XA$, XY, XO are coplanar. Why?

5. $\therefore YO$ lies in that same plane. § 330, 2

6. But $\because YO$ and $AX \perp OX$, § 339

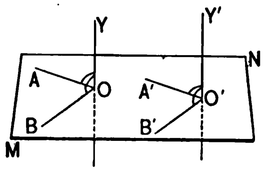
   $\therefore YO \parallel AX$, and similarly for all other $\perp$s.

   I, prop. XVI, cor. 3

Corollary. *From a point outside of a plane, not more than one line can be drawn perpendicular to that plane.*

PROPOSITION IX.

342. Theorem. *If one of two parallel lines is perpendicular to a plane, the other is also.*



Given $OY \parallel O'Y'$, $OY \perp$ plane MN at O, and $O'Y'$ meeting plane MN at O'.

To prove that $O'Y' \perp MN$.

Proof. 1. Let OA, OB be any lines from O, in MN, $O'A' \parallel OA$, and $O'B' \parallel OB$.

2. Then $\angle s\ YOA$, YOB are rt. $\angle s$. § 339

3. But $\angle YOA = \angle Y'O'A'$, $\angle YOB = \angle Y'O'B'$. Prop. V

4. $\therefore \angle s\ Y'O'A'$, $Y'O'B'$ are also rt. $\angle s$, Prel. prop. I
and $O'Y' \perp MN$. § 339

343. Definitions. The **projection of a point** on a plane is the foot of the perpendicular through that point to the plane.

The **projection of a line** on a plane is the locus of the projections of all of its points.

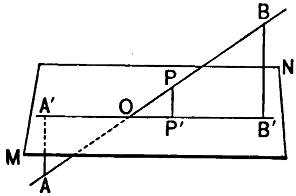
Exercises. 536. Are lines which make equal angles with a given line always parallel? (Answer by drawing figures to illustrate.)

537. Show how to determine the perpendicular to a plane, through a given point, by the use of two carpenter's squares.

538. Prove prop. VI on the following outline: Assume B on y, and draw ABC so that $AB = BC$ (How is this done?); prove $2 \cdot PB^2 + 2 \cdot BC^2 = PA^2 + PC^2 = 2 \cdot PO^2 + OC^2 + OA^2 = 2 \cdot PO^2 + 2 \cdot OB^2 + 2 \cdot BC^2$; $\therefore PB^2 = PO^2 + OB^2$; $\therefore \angle POB$ is a rt. $\angle$.

Proposition X.

344. Theorem. *The projection of a straight line on a plane is the straight line which passes through the projections of any two of its points.*



Given A', P', B', the projections of A, P, B, points in the line AB, on the plane MN.

To prove that P' is in the straight line $A'B'$.

Proof. 1. $AA' \parallel BB' \parallel PP'$. Why?

2. $\therefore A, A', B, B'$ are coplanar. Prop. I, cor. 2

3. $\therefore P$ is in that same plane. Why?

4. $\therefore PP'$ is in that same plane. Prop. I, cor. 3

5. $\therefore A', P', B'$ are collinear. Prop. II

Corollary. *If a line intersects a plane, its projection passes through the point of intersection.*

345. Definitions. The smallest angle formed by a line and its projection on a plane is called the **inclination** of the line to the plane or **the angle of the line and the plane.**

A **figure** is said to be **projected on a plane** when all of its points are projected on the plane.

The plane determined by a line and its projection on another plane is called the **projecting plane.**

In the figure of prop. X, $\angle B'OB$ is the *inclination* of AB to MN. The plane determined by AB, $A'B'$ is the *projecting plane.*

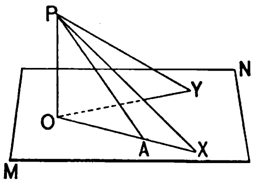
PROPOSITION XI.

346. Theorem. *Of all lines that can be drawn from a point to a plane,*

1. The perpendicular is the shortest;

2. Obliques with equal inclinations are equal, and conversely;

3. Obliques with equal projections are equal, and conversely.



1. Given $PO \perp$ plane MN, PX oblique to MN.

To prove that $PO < PX$.

Proof. $\because \angle XOP =$ rt. $\angle$, (Why?) $\therefore PO < PX$. I, prop. XX

2. Given $PO \perp MN, \angle PYO = \angle PXO$.

To prove that $PY = PX$, which is true because $\triangle POY \cong \triangle POX$. I, prop. XIX, cor. 7

CONVERSELY: **Given** $PO \perp MN$, $PY = PX$.

To prove that $\angle PYO = \angle PXO$, which is true because $\triangle POY \cong \triangle POX$. I, prop. XIX, cor. 5

3. Given $PO \perp MN$, $OY = OX$.

To prove that $PY = PX$, which is true because $\triangle POY \cong \triangle POX$. I, prop. I

CONVERSELY: **Given** $PO \perp MN$, $PY = PX$.

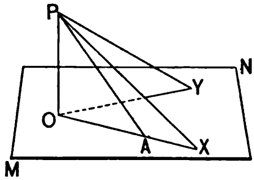
To prove that $OY = OX$, which is true because $\triangle POY \cong \triangle POX$. I, prop. XIX, cor. 5

PROPOSITION XII.

347. Theorem. *From a point to a plane,*

1. Of two obliques with unequal inclinations, that having the greater inclination is the shorter, and conversely;

2. Of two obliques with unequal projections, that having the longer projection is the longer, and conversely.



1. Given $PO \perp MN$, PY and PA two obliques such that $\angle PAO > \angle PYO$.

To prove that $PA < PY$.

Proof. 1. Suppose X taken on OA so that $OX = OY$.

2. Then $\triangle POX \cong \triangle POY$, and $PX = PY$. Why?

3. But PX, and $\therefore$ its equal PY, $> PA$. I, prop. XX

CONVERSELY:

Given $PO \perp MN$, PY and PA two obliques such that $PA < PY$.

To prove that $\angle PAO > \angle PYO$.

Proof. 1. Suppose X taken on OA so that $OX = OY$.

2. Then $\triangle POX \cong \triangle POY$, $\angle PXO = \angle PYO$, and $PX = PY$. Why?

3. $\therefore PA < PX$, $\because PA < PY$. Given

4. X cannot fall on A, for then $PA \equiv PX$.

5. Nor between O and A, for then $PA > PX$. Why?

6. $\therefore X$ is on OA produced,

and $\therefore \angle PAO > \angle PXO$. I, prop. V

7. $\therefore \angle PAO > \angle PYO$. Subst.

2. Given $PO \perp MN$, PY and PA two obliques such that $OA < OY$.

To prove that $PA < PY$.

Proof. 1. Suppose X taken on OA so that $OX = OY$.

2. Then $\triangle POX \cong \triangle POY$, and $PX = PY$. I, prop. I

3. And $\because OA < OY$, or OX,

$\therefore PA < PX$, or PY. Why?

CONVERSELY:

Given $PO \perp MN$, PY and PA two obliques such that $PA < PY$.

To prove that $OA < OY$.

Proof left for the student.

Definition. The length of the perpendicular from a point to a plane is called the **distance** from that point to the plane.

E.g. in the figure on p. 258, the distance from P to MN is the length of PO.

Exercises. 539. Prove that if three concurrent lines meet a fourth line, not in the same point, the four lines are coplanar.

540. Why does folding a sheet of paper give a straight edge?

541. Suppose it known that a point P is in each of the three planes X, Y, Z. Is P probably fixed? Is it necessarily fixed?

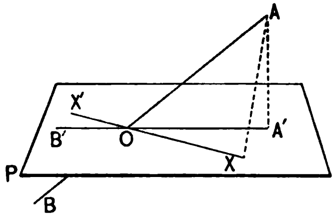
542. If the triangles ABC, $A'B'C'$, in different planes, are such that AB and $A'B'$ meet when produced, as also BC and $B'C'$, and CA and $C'A'$, then the lines AA', BB', CC' are either concurrent or parallel.

543. How many planes are determined by n concurrent lines, no three of which are coplanar?

544. If a line cuts one of two parallel lines, must it cut the other? If it does, are the corresponding angles equal?

Proposition XIII.

348. Theorem. ***The acute angle which a line makes with its own projection on a plane is the least angle which it makes with any line in that plane.***



Given the line AB, cutting plane P at O, $A'B'$ the projection of AB on P, and XX' any other line in P, through O.

To prove that $\angle A'OA < \angle XOA$.

Proof. 1. Suppose A' the projection of A, OX made equal to OA', and AX, AA' drawn.

2. Then $AA' < AX$. Prop. XI, 1

3. $\therefore$ in $\triangle$ OXA and $OA'A$, we have

$OX = OA'$, $OA \equiv OA$, and $AA' < AX$,

$\therefore \angle A'OA < \angle XOA$. I, prop. XI

Exercises. 545. The obtuse angle which a line makes with its own projection (produced) on a plane is the greatest angle which it makes with any line in that plane.

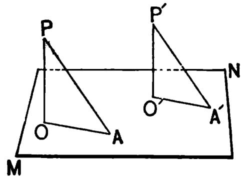
546. In a given plane, to determine the locus of points equidistant from two fixed points in space.

547. Prove prop. IX by supposing $O'Y'$ not perpendicular to MN, but supposing another line $O'Z$ from $O' \perp$ to MN; then prove that $O'Z$ would be parallel to OY, which would violate § 85, and hence be absurd.

548. As a special case of prop. X, suppose $AB \perp MN$; what would be its projection and its inclination?

PROPOSITION XIV.

349. Theorem. *Parallel lines intersecting the same plane are equally inclined to it.*



Given two parallels, PA, $P'A'$, intersecting a plane MN at A, A'; and O, O' the projections of P, P'.

To prove that $\angle PAO = \angle P'A'O'$.

Proof. 1. $\because PO$ and $P'O' \perp MN$,

$\therefore PO \parallel P'O'$. Why?

2. $\therefore \angle OPA = \angle O'P'A'$. Prop. V

(Let the student complete the proof.)

350. Definition. Two straight lines, not coplanar, are regarded as forming an **angle** which is equal to the one formed by either line and a line drawn, from a point upon it, parallel to the second.

E.g. in the figure of prop. XIV, the angle made by AO and $P'A'$ is considered as $\angle P'A'O'$ or $\angle PAO$.

Exercises. 549. Parallel line-segments are proportional to their projections on a plane.

550. In general, which is the longer, a line-segment or its projection? Is there any exception?

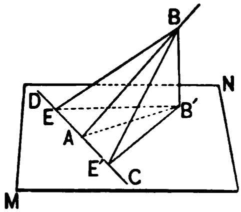
551. Show how, with a 10 ft. pole marked in feet, to determine the foot of the perpendicular let fall to the floor from the ceiling of a room 8 ft. high.

552. Show how a line 1 in. long and another 2 in. long may have equal projections on a plane.

553. If any two lines are parallel, respectively, to two others, an angle made by the first pair equals one made by the second.

Proposition XV.

351. Theorem. *If a line intersects a plane, the line in the plane perpendicular to the projection of the first line at the point of intersection is perpendicular to the line itself.*



Given AB intersecting the plane MN at A, B' the projection of B on MN, and $DC \perp AB'$ at A.

To prove that $DC \perp AB$.

Proof. 1. On DC let $EA = AE'$; join E, E' to B and B'.

2. Then $\triangle AB'E \cong \triangle AB'E'$, and $EB' = E'B'$. Why?

3. Then $\triangle EBB' \cong \triangle E'BB'$, and $EB = E'B$. Why?

4. Then $\triangle E'AB \cong \triangle EAB$, and $\angle E'AB = \angle BAE$. Why?

5. $\therefore DC \perp AB$, by defs. of rt. $\angle$ and $\perp$.

352. Definition. A line is said to be **parallel to a plane** when it never meets the plane, however far produced. In that case, also, the plane is said to be **parallel to the line.**

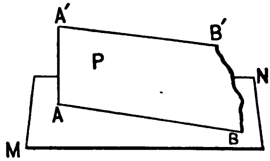
Exercises. 554. Prove prop. XV on the following outline: draw through B' a line $\parallel$ to DC; prove this parallel perpendicular to plane $AB'B$; $\therefore DC \perp$ plane $AB'B$, $\therefore DC \perp AB$.

555. Prove prop. XV by showing that $AE'^2 + AB^2 = BE'^2$, and that therefore $\angle E'AB$ is right.

556. In the figure of prop. XV prove that the area and perimeter of $\triangle AB'B$ are respectively less than those of $\triangle EB'B$.

PROPOSITION XVI.

353. Theorem. *Any plane containing only one of two parallel lines is parallel to the other.*



Given the parallel lines AB, $A'B'$, and the plane MN containing AB but not $A'B'$.

To prove that $MN \parallel A'B'$.

Proof. 1. AB and $A'B'$ determine a plane P. Prop. I, cor. 2

2. $\because$ AB and $A'B'$ lie wholly in P, $\therefore$ if $A'B'$ meets MN it meets AB. § 85

3. But $\because$ $AB \parallel A'B'$, this is impossible. § 82, def. $\parallel$ lines

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Exercises. 557. A line which is parallel to a plane is parallel to its projection on that plane.

558. Through a point without a straight line any number of planes can pass parallel to that line.

559. If a line is parallel to a plane, the intersection of that plane with any plane passing through that line is parallel to the line.

560. If from two points on a line parallel to a plane, parallel lines are drawn to and terminated by that plane, these parallel lines are equal.

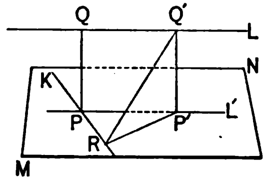
561. If a line is parallel to a plane, and if from any point in the plane a line is drawn parallel to the first line, then the second line lies wholly in the plane.

562. If, through a line parallel to a plane, several planes pass so as to intersect that plane, these lines of intersection are parallel.

563. If the distances from two given points on the same side of a plane, to that plane, are equal, the line determined by those points is parallel to the plane.

Proposition XVII.

354. Theorem. *Between two lines not in the same plane, one, and only one, common perpendicular can be drawn.*



Given two lines K, L, not coplanar.

To prove that one, and only one, common perpendicular can be drawn between them.

Proof. 1. Let MN be the plane, through K, $\parallel L$. (Can such a plane exist?) Let L' be the projection of L on MN.

2. Then K is not $\parallel$ to L', for then it would be $\parallel$ to L. Prop. IV

Let K intersect L' at P.

3. A $\perp$ to L and K is $\perp$ to MN. Why?

4. Then $\because L'$ is the locus of the feet of all $\perp$s from points in L, on plane MN, § 343, def. projection

$\therefore P$ is the unique point in which a $\perp$ from a point on L, to K, can meet K.

5. $\therefore$ if PQ is drawn $\perp$ to L, it is $\perp$, and the only $\perp$, to both L and K.

Corollary. *The common perpendicular is the shortest line-segment between two lines not in the same plane.*

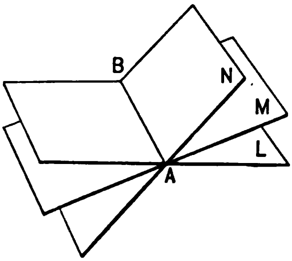
For if $Q'P' \parallel QP$, then $QP = Q'P' < Q'R$. Prop. XI, 1

355. Definition. The length of the common perpendicular from one line to another is called the **distance** between those lines.

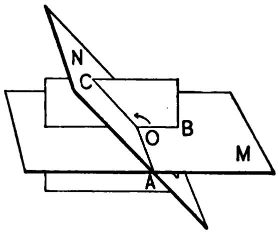
3. PENCIL OF PLANES.

356. Definitions. Any number of planes containing the same line are said to form a **pencil of planes**; the line is called its **axis.**

357. Any two planes of a pencil are said to form a **dihedral angle.**



***LMN*, a pencil of planes; *AB*, the axis of the pencil.**



Dihedral angles formed by the planes *M* and *N*. Dihedral angle *MN* measured by plane angle *BOC*. *AO* the edge of the dihedral angle.

The two planes of a dihedral angle are called the **faces,** and the axis of the pencil is called the **edge** of the dihedral angle.

Two intersecting planes form more than one dihedral angle, just as two intersecting lines form more than one *plane angle*, the latter term now being used to designate an angle made by lines in a plane.

358. A plane of a pencil turning about the axis from one face of a dihedral angle to the other is said to turn through the angle, *the angle being greater* as the amount of turning is greater.

Since the size of a dihedral angle depends only upon the amount of turning just mentioned, it is *independent of the extent of the faces.*

359. If perpendiculars are erected from any point in the edge of a dihedral angle, one in each face, the size of the plane angle thus formed evidently varies as the size of the dihedral angle. Hence a dihedral angle is said to be *measured* by that plane angle, or, strictly, to have the same numerical measure.

360. A dihedral angle is said to be *acute*, *right*, *obtuse*, *oblique*, *reflex*, *straight*, according as the measuring plane angle is so, and it is usually named by its measuring plane angle, or merely by its faces in counter-clockwise order.

The terms *adjacent angles*, *bisector*, *sum and difference of dihedral angles*, *point within or without the angle*, *complement*, *supplement*, *conjugate*, and *vertical angles* will readily be understood from the corresponding terms in plane geometry.

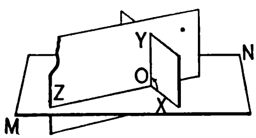
As with plane angles the smallest angle made by two intersecting lines is, in general, to be understood unless the contrary is stated, so with dihedral angles.

If a dihedral angle is right, the planes are said to be **perpendicular** to each other.

E.g. in the following figure, $ZY \perp MN$.

Proposition XVIII.

361. Theorem. *If a line is perpendicular to a plane, any plane containing this line is also perpendicular to that plane.*



Given OY perpendicular to the plane MN, and ZY any plane containing OY.

To prove that $ZY \perp MN$.

Proof. 1. Suppose OX, in MN, $\perp$ OZ, the intersection of MN and ZY.

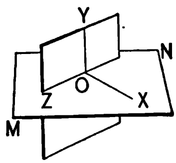
Then $\angle$s YOZ, XOY are right $\angle$s. Why?

2. But $\because$ $\angle$ XOY fixes the measure of the dihedral $\angle$, § 359

$\therefore$ $ZY \perp MN$. Def.

PROPOSITION XIX.

362. Theorem. *If two planes are perpendicular to each other, any line in one of them, perpendicular to their intersection, is perpendicular to the other.*



Given the planes $ZY \perp MN$, OZ their intersection, and OX, in MN, $\perp$ OZ.

To prove that $OX \perp ZY$.

Proof. 1. Let OY, in ZY, be $\perp$ to OZ at O.

Then $\angle$ XOY is the measuring angle. § 359

2. $\therefore$ $\angle$ XOY is right. Def. $\perp$ planes

3. But $\because$ $\angle$ ZOX is also right, Why?

$\therefore$ $OX \perp ZY$. Why?

COROLLARIES. 1. *If two planes are perpendicular to each other, a line from any point in their line of intersection, perpendicular to either, lies in the other.*

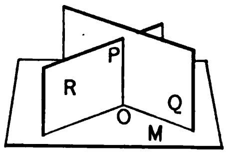
By the theorem, $OY \perp MN$, and it lies in ZY; and by prop. VII, cor. 2, only one perpendicular to MN can be drawn from O.

2. *Through a point without a line not more than one plane can pass perpendicular to that line.*

For if through Y another plane could pass $\perp$ OX, it would pass through O, $\because$ $\angle XOY =$ rt. $\angle$, and only one $\perp$ can be drawn from Y to OX. But the plane would also include line OZ, else there would be two $\perp$s from O to OX in the plane MN.

Proposition XX.

363. Theorem. *If each of two intersecting planes is perpendicular to a third plane, their line of intersection is also perpendicular to that plane.*



Given two planes, Q, R, intersecting in OP, and each perpendicular to plane M.

To prove that $OP \perp M$.

Proof. 1. A $\perp$ to M from O lies in Q and in R. Prop. XIX, cor. 1

2. $\therefore$ it coincides with OP, the only line common to Q and R. $\therefore$ $OP \perp M$.

Exercises. 564. To construct a plane containing a given line, and parallel to another given line. (Assumed in step 1 of prop. XVII.)

565. Prove that vertical dihedral angles are equal.

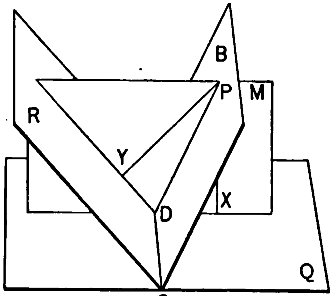
566. How many degrees in the measure of the dihedral angle between the plane of the earth's equator and the ecliptic?

567. Prove that the edge of a dihedral angle is perpendicular to the plane of the measuring angle.

568. Prove that a line and its projection on a plane determine a second plane perpendicular to the first.

PROPOSITION XXI.

364. Theorem. *Any point in a plane which bisects a dihedral angle is equidistant from the faces of the angle.*



Given a dihedral angle, with faces Q, R, and edge CD, bisected by plane B; P, any point in B, with $PX \perp Q$, $PY \perp R$.

To prove that $PX = PY$.

Proof. 1. Let M be the plane of PX, PY, and D its intersection with CD.

2. Then $\because PX \perp Q$, $\therefore M \perp Q$. Why?

3. Similarly, $M \perp R$. Why?

4. $\therefore M \perp CD$. Prop. XX

5. $\therefore CD \perp DX, DY, DP$, whose $\measuredangle$s therefore measure the dihedral $\measuredangle$s. § 359

6. $\therefore \angle XDP = \angle PDY$.

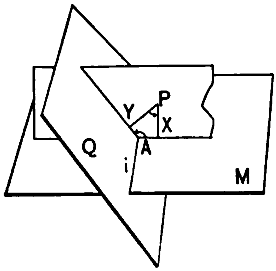
And $\because \angle X = \angle Y =$ rt. $\angle$, and $DP \equiv DP$,

7. $\therefore \triangle DXP \cong \triangle DYP$, and $PX = PY$. § 88, cor. 7

COROLLARY. *The locus of points that are equidistant from two intersecting planes is the pair of planes bisecting their dihedral angles.*

Proposition XXII.

365. Theorem. *If from any point lines are drawn perpendicular to two intersecting planes, the angle formed by these perpendiculars has a measure equal or supplemental to that of the dihedral angle of the planes.*



Given the planes M, Q, intersecting in i; lines $PX \perp M$, $PY \perp Q$; and plane PYX cutting i at A.

To prove that $\angle YPX$ is equal or supplemental to the dihedral angle MQ.

Proof. 1. Plane $YXP \perp M$, also $\perp Q$. Prop. XVIII

2. $\therefore$ plane $YXP \perp i$. Prop. XX

3. $\therefore XA$ and $YA \perp i$. § 339

4. $\therefore \angle XAY$ measures dihedral $\angle MQ$. § 359

5. But $\because \angle X = \angle Y =$ rt. $\angle$, Given

$\therefore \angle YPX$ is supplemental to $\angle XAY$, or dihedral $\angle MQ$. I, prop. XXI, cor.

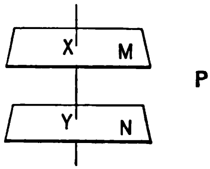
Corollary. *If the point is within the dihedral angle, the angles are supplemental.*

Definition. If two planes do not meet, however far produced, they are said to be **parallel**.

The term *pencil of parallels* is applied to planes as well as to lines.

PROPOSITION XXIII.

366. Theorem. *Planes perpendicular to the same straight line are parallel*



Given two planes, M, N; $\perp$ line XY, at X, Y, respectively.

To prove that $M \parallel N$.

Proof. If M and N should meet, as at P, then two planes would pass through $P \perp XY$, which is impossible.
Prop. XIX, cor. 2

Exercises. 569. Prove that through a point without a plane any number of lines can pass parallel to the plane.

570. Problem : To bisect a dihedral angle.

571. To find the locus of points equidistant from two fixed planes, and equidistant from two fixed points.

572. To find a point equidistant from two given planes, equidistant from two given points, and also at a given distance from a third plane.

573. Prove prop. XXII for the case in which the point P is taken in plane M.

574. In the figure on p. 270, as $\angle XAY$ increases from zero to a straight angle, what change does $\angle YPX$ undergo ?

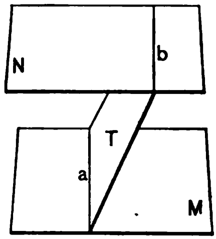
575. Also, suppose $\angle XAY = 120°$; what angle will PY make with plane M, if produced through Q to M?

576. Given two points, V, W, in two intersecting planes, M, Q, respectively. Find Z in the line of intersection of M and Q, such that $VZ + ZW$ shall be a minimum.

577. If from two points on a line parallel to a plane, parallel lines are drawn to that plane, a parallelogram is formed.

PROPOSITION XXIV.

367. Theorem. *The lines in which two parallel planes intersect a third plane are parallel.*



Given two parallel planes, M, N, intersected by a third plane, T, in lines a, b.

To prove that $a \parallel b$.

Proof. 1. a and b are in the same plane T.

2. And they cannot meet, because they are in M and N respectively, and $M \parallel N$.

3. $\therefore$ they are parallel by definition.

COROLLARY. *A line perpendicular to one of two parallel planes is perpendicular to the other.*

Pass two planes through that line and apply prop. XXIV and the def. of a plane $\perp$ to a line.

Exercises. 578. Through a given point only one plane can pass parallel to a given plane.

579. If two parallel planes intersect two other parallel planes, the four lines of intersection are parallel.

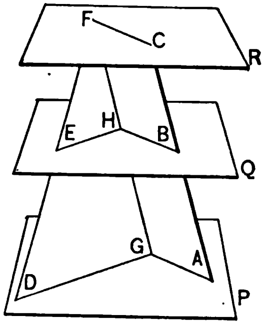
580. Parallel lines have parallel projections on any plane. (Suppose, as a special case, that the lines are perpendicular to the plane.)

581. If two lines are at right angles, are their projections on any plane also at right angles?

582. If two planes are perpendicular to each other, any line perpendicular to one of them is either parallel to or lies in the other.

PROPOSITION XXV.

368. Theorem. *If two straight lines are cut by parallel planes, the corresponding segments are proportional.*



Given ABC, DEF, two lines cut by planes P, Q, R, in points A, B, C and D, E, F.

To prove that $AB : BC = DE : EF$.

Proof. 1. Suppose the line GHF, drawn through F, $\parallel ABC$, cutting P, Q at G, H, respectively.

Then AC, GF determine a plane; also DF, GF. Prop. I, cors. 2, 1

2. $\therefore AG \parallel BH \parallel CF$, and $DG \parallel EH$. Why?

3. $\therefore AB = GH$, and $BC = HF$. I, prop. XXIV

4. But $\because GH : HF = DE : EF$, IV, prop. X, cor. 1

$\therefore AB : BC = DE : EF$. Subst. 3 in 4

Exercises. 583. In a gymnasium swimming tank the water is 5 ft. deep, and the ceiling is 9 ft. above the water; a pole 18 ft. long rests obliquely on the bottom of the tank and touches the ceiling. How much of the pole is in the water?

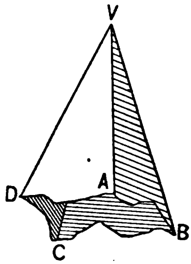
584. In the figure of prop. XXV, connect C and D, and prove the theorem without using the line FG.

4. POLYHEDRAL ANGLES.

369. Definitions. When a portion of space is separated from the rest by three or more planes which meet in but one point, the planes are said to form, or to include, a **polyhedral angle.**

A polyhedral angle is also called a **solid angle.**

As two intersecting lines form an infinite number of plane angles, but the smallest is considered unless the contrary is stated, and similarly with two intersecting planes, so three or more intersecting planes form an infinite number of polyhedral angles, but, as with plane and dihedral angles, only the smallest is considered.



A polyhedral angle, *V-ABCD*. *V*, the vertex; *VA*, *VB*, *VC*, *VD*, the edges; planes *VAB*, *VBC*, ..., the faces.

The lines of intersection of the planes of a polyhedral angle, each with the next, are called the **edges** of the polyhedral angle.

On account of the complexity of the general figure, the planes which form a polyhedral angle are considered as cut off by the edges, as in the above figure. So also the edges, which may be produced indefinitely, are considered as cut off by the vertex unless the contrary is stated.

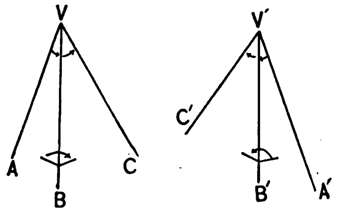
The portions of the planes which form a polyhedral angle, limited by the edges, are called the **faces** of the angle.

370. Polyhedral angles contained by 3, 4, ..., n planes are termed respectively **trihedral, tetrahedral,** ..... ***n*-hedral** angles.

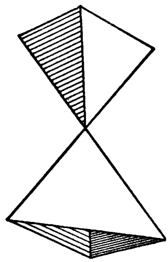
A polyhedral angle is specifically designated by a letter at its vertex, or by that letter followed by a hyphen, and letters on the successive edges.

371. Congruent polyhedral angles are such as have their dihedral angles equal, and the plane angles of their faces also equal and arranged in the same order.

372. Symmetric polyhedral angles are such as have their dihedral angles equal, and the plane angles of their faces also equal, but arranged in reverse order.



Symmetric polyhedral angles.



Opposite polyhedral angles.

Thus, in the above figure, V and V' are symmetric trihedral angles, the letters showing the reverse arrangement.

Some idea of this reverse arrangement may be obtained by thinking of two gloves, fitting the right and left hands respectively. As two such gloves are not congruent, so, in general, two symmetric polyhedral angles are not congruent.

373. Opposite polyhedral angles are such that each is formed by producing the edges and faces of the other through the vertex.

Exercises. 585. How many edges in an n-hedral angle? How many dihedral angles? How many plane face angles? How many vertices?

586. If a straight line is oblique to one of two parallel planes, it is to the other.

587. If a plane intersects all the faces of a tetrahedral angle, what kind of a plane figure is formed by the lines of intersection? What, in the case of a trihedral angle?

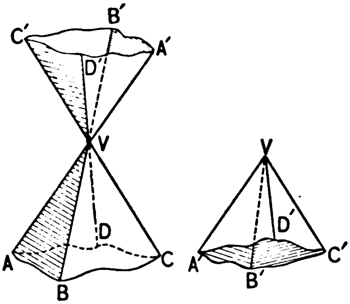
588. Does the magnitude of a polyhedral angle depend upon the lengths of the edges?

589. Construct from stiff paper two symmetric trihedral angles, with face angles of about 30°, 60°, 45°, and see if they are congruent. (No proof required.)

590. If each of two intersecting lines is parallel to a plane, so is the plane of those lines.

Proposition XXVI.

374. Theorem. *Opposite polyhedral angles are symmetric.*



Given $V\text{-}ABCD$, any polyhedral angle, and $V\text{-}A'B'C'D'$, its opposite polyhedral angle.

To prove that $V\text{-}ABCD$ and $V\text{-}A'B'C'D'$ are symmetric.

Proof. 1. $\angle AVB = \angle A'VB'$,

$\angle BVC = \angle B'VC'$, ... Prel. prop. VI

2. Dihedral $\angle$s with edges VB, VB', being formed by the same planes, have equal (vertical) measuring angles. Prel. prop. VI

3. So for the other dihedral $\angle$s. But the order of arrangement in the one is reversed in the other.
$\therefore$ the polyhedral $\angle$s are symmetric.

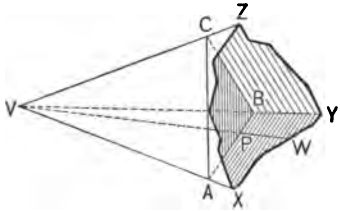
Note. That the order of the angles is reversed appears more clearly to the eye by making two opposite trihedral angles of pasteboard. It is also seen by tipping the upper angle over, as has been done in the figure to the right.

Exercises. 591. If the edges of one polyhedral angle are respectively perpendicular to the faces of a second polyhedral angle, then the edges of the latter are respectively perpendicular to the faces of the former.

592. Two parallel planes intersecting two parallel lines cut off equal segments.

PROPOSITION XXVII.

375. Theorem. *In any trihedral angle the sum of any two face-angles is greater than the third.*



Given the trihedral angle $V\text{-}XYZ$.

To prove that $\angle YVZ + \angle ZVX > \angle XVY$.

Proof. 1. If $\angle XVY \not> $ either $\angle YVZ$ or $\angle ZVX$, no proof is necessary. Why not?

2. If $\angle XVY >$ either $\angle YVZ$ or $\angle ZVX$, suppose it $> \angle ZVX$.

3. Then in plane VXY suppose VW drawn, making $\angle XVW = \angle ZVX$.

Suppose VC taken on VZ, equal to VP on VW, and a plane passed through C, P, and any point A of VX. Let this plane intersect VY at B.

4. Then $\triangle AVP \cong \triangle AVC$, and $AC = AP$. I, prop. I

5. But $\because AC + CB > AB$, or $AP + PB$, Why?

$\therefore CB > PB$. Why?

6. $\therefore$ in $\triangle$s PVB and CVB, $\angle BVC > \angle PVB$.
I, prop. XI

7. $\therefore \angle CVA + \angle BVC > \angle AVP + \angle PVB$, or $\angle AVB$.

Or $\angle YVZ + \angle ZVX > \angle XVY$. Ax. 4

Corollaries. 1. *In any trihedral angle the difference of any two face-angles is less than the third.*

For if the face-angles are a, b, c, then since $a + b > c$, $\therefore a > c - b$.

2. *In any polyhedral angle any face-angle is less than the sum of all the other face-angles.*

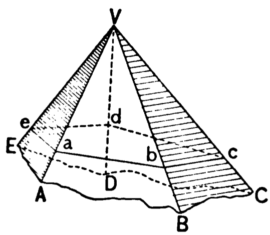
For the polyhedral angle may be divided into a number of trihedral angles, and prop. XXVII repeatedly applied.

Note. Prop. XXVII and corollaries suppose that each face-angle is less than a straight angle. This is in accordance with the note under the definition of a polyhedral angle.

376. Definition. A polyhedral angle is said to be **convex** when any polygon formed by a plane cutting every face, is convex; otherwise it is said to be **concave.**

Proposition XXVIII.

377. Theorem. *In any convex polyhedral angle the sum of the face-angles is less than a perigon.*



Given any convex polyhedral angle, $V\text{-}ABC$ .....

To prove that $\angle AVB + \angle BVC + \angle CVD + \cdots\cdots < \text{perigon}$.

Proof. 1. Let the faces of the angle be cut by a plane. This will form a convex polygon of n sides ($n = 5$ in the figure), abc ..... Def. convex polyh. $\angle$

Let $S_v =$ sum of plane $\angle$s aVb, bVc, ..., at the vertex;

$S_b =$ sum of plane $\angle$s baV, Vba, cbV, ..., at the bases of the $\triangle$s;

and $S_p =$ sum of plane $\angle$s cba, dcb, ..., of the polygon.

2. Then $S_p = (n-2)$ st. $\angle$s, or $S_p + 2$ st. $\angle$s $= n$ st. $\angle$s. I, prop. XXI

3. And $S_v + S_b = n$ st. $\angle$s, since there is a st. $\angle$ for each $\triangle$. I, prop. XIX

4. $\therefore S_v + S_b = S_p +$ perigon. Steps 2 and 3; ax. 1

5. And $\because S_b > S_p$, Prop. XXVII
$\therefore S_v <$ perigon.

Exercises. **593.** The three planes which bisect the three dihedral angles of a trihedral angle intersect in a common line whose points are equidistant from the three faces. (See prop. XXI, cor., and I, prop. XLIV.)

594. Suppose a polyhedral angle formed by three, four, five equilateral triangles. What is the sum of the face-angles at the vertex?

595. If lines through any point O and the vertices, A, B, C, ..., of a polygon, cut a plane parallel to the plane of that polygon in A', B', C', ..., prove that $A'B'C'$ ..... $\sim ABC$ ..... and that the ratio of similitude is that of OA' to OA.

596. In ex. 595, the more remote O is from the planes ABC ....., $A'B'C'$ ....., the more nearly do AA', BB', CC' ..... become parallel; suppose they become parallel, state and prove the resulting theorem.

597. In ex. 595, if plane $A'B'C'$ ..... were not parallel to plane ABC ....., prove that the corresponding sides, AB, $A'B'$, and BC, $B'C'$, and CD, $C'D'$, ..., would, in general, meet in points on the intersection of the two planes.

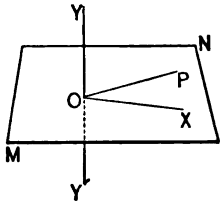
598. Two planes, each parallel to a third plane, are parallel to each other.

599. Ex. 598 is analogous to I, prop. XVIII. State the theorem and corollaries analogous to I, prop. XVII and its corollaries, and prove them.

5. PROBLEMS.

PROPOSITION XXIX.

378. Problem. *Through a given point to pass a plane perpendicular to a given line: (1) the point being without the line, (2) the point being on the line.*



1. Given the line YY', and point P without.

Required through P to pass a plane $\perp$ YY'.

Construction. 1. From P draw $PO \perp YY'$. I, prop. XXX

2. From O draw another line $OX \perp YY'$. I, prop. XXIX

Then MN, the plane of OP, OX, is the required plane.

Proof. $\because YY' \perp OP$,

and $YY' \perp OX$,

$\therefore YY' \perp MN$. § 339

2. Given the line YY', and the point O upon it.

Required through O to pass a plane $\perp$ YY'.

Construction and Proof. Draw OP and $OX \perp YY'$. This can be done because the three lines are not required to be coplanar.

Then the plane $XOP \perp YY'$. § 339

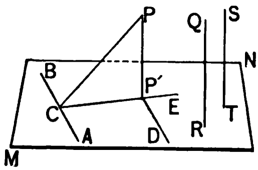
PROPOSITION XXX.

379. Problem. *Through a given point to pass a plane parallel to a given plane.*

Solution. Draw two intersecting lines in the given plane. Through the given point draw two lines parallel to these lines, thus determining the required plane.

PROPOSITION XXXI.

380. Problem. *Through a given point to draw a line perpendicular to a given plane: (1) the point being without the plane, (2) the point being in the plane.*



1. Given the plane MN, and the point P without.

Required to draw a perpendicular from P to MN.

Construction. 1. Draw $PC \perp AB$, any line in MN. I, prop. XXX

2. In MN draw $CE \perp AB$. I, prop. XXIX

3. Draw $PP' \perp CE$. I, prop. XXX

Then PP' is the required perpendicular.

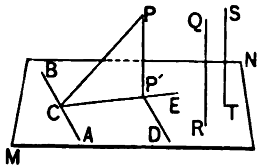
Proof. 1. $CA \perp$ plane CPP'. Prop. VI, cor. 1

2. Draw $P'D \parallel CA$; then $P'D \perp$ plane CPP'. Prop. IX

3. $\therefore \angle DP'P$ is right, and $PP' \perp P'D$. § 339

4. But $PP' \perp CP'$, and $\therefore PP' \perp MN$. Prop. VI, cor. 1

2. Given the plane MN, and the point R within it.



Required through R to draw a perpendicular to MN.

Construction. 1. From any external point S draw $ST \perp MN$. Case 1

2. From R draw $RQ \parallel TS$. I, prop. XXXIII

Proof. Then RQ is the required perpendicular. Why ?

Exercises. **600.** From the point of intersection of two lines to draw a line perpendicular to each of them.

601. To determine the point whose distances from the three faces of a given trihedral angle are given. Is it unique ?

602. From the vertex of a trihedral angle to draw a line making equal angles with the three edges.

603. The three planes, through the bisectors of the face-angles of a trihedral angle, perpendicular to those faces, intersect in a common line whose points are equidistant from the edges. (See I, prop. XLIII.)

604. In how many ways can a polyhedral angle be formed with equilateral triangles and squares ?

605. Prove that a straight line makes equal angles with parallel planes.

606. If each of two intersecting planes is parallel to a given line, prove that their intersection is coplanar with that line.

607. Prove that parallel lines make equal angles with parallel planes.

608. Are planes perpendicular to the same plane parallel ?

609. In the figure of prop. XXV, without drawing FG, draw CD and AF; then show that the four lines CD, CA, FD, FA intersect plane Q in the vertices of a parallelogram.

610. Given two lines, not coplanar, and a plane not containing either line, required to draw a straight line which shall cut both given lines and shall be perpendicular to the plane. (Project both lines on the plane.)

BOOK VII. — POLYHEDRA.

I. GENERAL AND REGULAR POLYHEDRA.

381. Definitions. A solid whose bounding surface consists entirely of planes is called a **polyhedron**; the polygons which bound it are called its **faces**; the sides of those polygons, its **edges**; and the points where the edges meet, its **vertices.**

382. If a polyhedron is such that no straight line can be drawn to cut its surface more than twice, it is said to be **convex**; otherwise it is said to be **concave.**

Unless the contrary is stated, the word *polyhedron* means convex polyhedron. The word *convex* will, however, be used wherever necessary for special emphasis.

383. If the faces of a polyhedron are congruent and regular polygons, and the polyhedral angles are all congruent, the polyhedron is said to be **regular.**

Exercises. 611. Draw a figure of a polyhedron of four faces. Count the edges, faces, and vertices, and show that the number of edges plus two equals the number of faces plus the number of vertices.

612. Do the same for a polyhedron of five faces; also of six faces.

613. Take a piece of chalk, apple, or potato, and see if a seven-edged polyhedron can be cut from it.

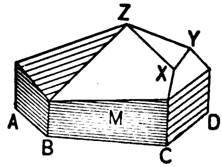
614. What is the locus of points on the surface of a polyhedron equidistant from two given vertices? (The distances are to be taken as usual on a straight line, and not necessarily on the surface.)

615. What is the locus of points equidistant from two given non-parallel faces of a given polyhedron?

616. To find a point equidistant from two given vertices of a polyhedron, and from two given non-parallel faces.

PROPOSITION I.

384. Theorem. *If a convex polyhedron has* e *edges*, v *vertices, and* f *faces, then* e + 2 = f + v.



Given $ABC \ldots\ldots Z$, a convex polyhedron of e edges, v vertices, f faces.

To prove that $$e + 2 = f + v.$$

Proof. 1. Imagine $ABC \ldots\ldots Z$ formed by adding adjacent faces, beginning with any face as $ABCD \ldots\ldots$ of a sides, then adding face M, of b sides, and so on.

(It is advisable to build up a rectangular box of paste-board while reading the proof.)

Let e_r = the number of edges, and v_r = the number of vertices, after r faces have been put together.

(Thus when we put 2 rectangles together in building up the box, we have located 7 edges and 6 vertices; *i.e.* $e_2 = 7$, $v_2 = 6$, in this case.)

2. Then, since the first face had a sides, $\therefore e_1 = a$ and $v_1 = a$.

(In the box, $e_1 = 4$, $v_1 = 4$.)

3. $\because$ adding an adjacent face M, of b sides, gives only $(b - 1)$ new edges, and $(b - 2)$ new vertices (Why?),

(In the box, adding a second rectangle to the first gives only 3 new edges because we have 1 in common with the first face, and 2 new vertices because we have 2 in common with the first face.)

$\therefore e_2 = a + b - 1$, $v_2 = a + b - 2$, so that $e_2 - v_2 = 1$.

(In the box, $e_2 = 4 + 4 - 1 = 7$, $v_2 = 4 + 4 - 2 = 6$, so that $e_2 - v_2 = 1$.)

4. Similarly, if the next face has c sides, and the next d, etc.,

$$\begin{array}{rll} & e_3 = (a+b-1)+(c-2) & \text{or } a+b+c-3 \\ \text{and} & v_3 = (a+b-2)+(c-3) & \text{"} \quad \underline{a+b+c-5} \\ \therefore & e_3 - v_3 = & \qquad\qquad\qquad 2 \end{array}$$

(In the box, $e_3 = 9$, $v_3 = 7$, $\therefore e_3 - v_3 = 2$.)

Likewise

$$\begin{array}{rl} & e_4 = (a+b+c-3)+(d-3) \\ & v_4 = (a+b+c-5)+(d-4) \\ \therefore & e_4 - v_4 = \qquad\quad 2 + \qquad 1 = 3. \end{array}$$

(In the box, $e_4 = 10$, $v_4 = 7$, $\therefore e_4 - v_4 = 3$.)

And in general, $e_r - v_r = r - 1$.

5. But the addition of the last, or fth face, as XYZ, after all the others have been put together, gives no new edges or vertices,

$$\therefore e_f - v_f = e_{f-1} - v_{f-1} = f - 2.$$

(In the box, adding the last face merely puts on the cover, adding no new edges or vertices. $\therefore e_6 - v_6 = e_5 - v_5 = 4$, which is evidently true, because 12, the number of edges, minus 8, the number of vertices, is 4.)

6. That is, $e - v = f - 2$, so that $e + 2 = f + v$; for $e_f = e$, and $v_f = v$.

COROLLARY. *For every polyhedron there is another which, with the same number of edges, has as many faces as the first has vertices, and as many vertices as the first has faces.*

For in the equation $e + 2 = f + v$, the f and v may be interchanged without affecting the e.

NOTE. This theorem is known as Euler's, although Descartes knew and employed it. The theorem is very useful in the study of crystals.

Exercise. 617. If the faces of a polyhedron are all triangular, the number of faces is even and is four less than twice the number of vertices. (Since there are 3 edges to every face, but each edge belongs to 2 adjacent faces, $e = 3f/2$; substitute in $e + 2 = f + v$.)

Proposition II.

385. Theorem. *There cannot be more than five regular convex polyhedra.*

Proof. 1. Let n = number of sides in one face, and a = number of degrees in each plane $\angle$ of the faces of a regular convex polyhedron.

Then $a = (n-2) \cdot 180°/n$, I, prop. XXI

and if $n = 6$, then $a = 120°$, and $3\,a = 360°$.

2. $\therefore$ if $n = 6$ or more, there can be no solid angle. VI, prop. XXVIII

3. And if $n = 5$, then $a = 108°$, and $3\,a = 324°$,

and $\therefore$ 3 regular pentagons, but no more, can form a solid angle. VI, prop. XXVIII

4. And if $n = 4$, then $a = 90°$, $3\,a = 270°$, $4\,a = 360°$,

and $\therefore$ 3 squares, but no more, can form a solid angle. VI, prop. XXVIII

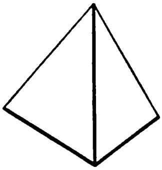
5. And if $n = 3$, $a = 60°$, $3\,a = 180°$, $4\,a = 240°$, $5\,a = 300°$, $6\,a = 360°$,

and $\therefore$ 3, 4, or 5 equilateral $\triangle$s, but no more, can form a solid angle. VI, prop. XXVIII

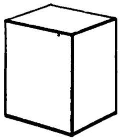
6. $\therefore$ there cannot be more than 5 regular convex polyhedra, viz. those formed by regular pentagons (3 at each vertex), squares (3 at each vertex), equilateral triangles (3, 4, or 5 at each vertex).

Note. There are five regular convex polyhedra; but the complete proof of the fact is not of enough importance to insert it in the body of the work. It may be given as an exercise, since it involves no new principles. These five polyhedra have been called the *Platonic Bodies*, from the attention given them in Plato's school, although they were known to the Pythagoreans. The three simpler forms enter largely into crystallography, usually somewhat modified.

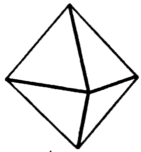
The five regular polyhedra are given on page 287.



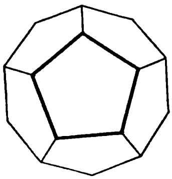
The regular *tetrahedron* (or triangular pyramid), formed by 4 equilateral triangles.



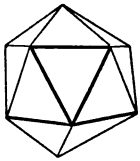
The regular *hexahedron* (or cube), formed by 6 squares.



The regular *octahedron*, formed by 8 equilateral triangles.

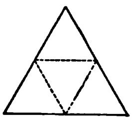


The regular *dodecahedron*, formed by 12 regular pentagons.

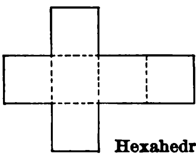


The regular *icosahedron*, formed by 20 equilateral triangles.

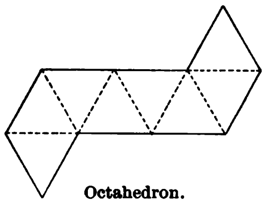
The five regular polyhedra can be constructed from cardboard by marking out the following, cutting through the heavy lines and half through the dotted ones, and then bringing the edges together.



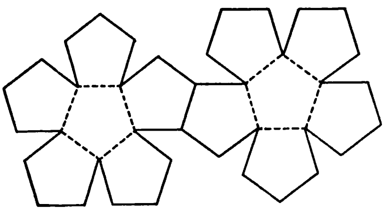
Tetrahedron.



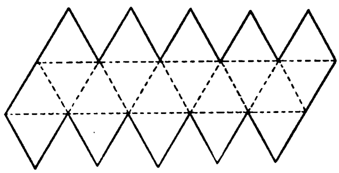
Hexahedron.



Octahedron.



Dodecahedron.



Icosahedron.

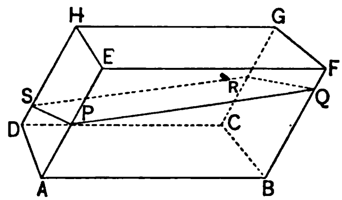
2. PARALLELEPIPEDS.

386. Definitions. A **Parallelepiped** is a solid bounded by three pairs of parallel planes.

The four lines through a parallelepiped, joining the opposite vertices, are called its **diagonals.**

Proposition III.

387. Theorem. *The opposite faces of a parallelepiped are congruent parallelograms; and any section of it, made by a plane cutting two pairs of opposite faces without cutting the remaining pair, is a parallelogram.*



Given the parallelepiped AG, and $PQRS$ a plane section cutting the parallel faces AF, DG, and AH, BG.

To prove (1) that AC and EG are congruent ▱s,
(2) that $PQRS$ is a ▱.

Proof. 1. $EF \parallel HG \parallel DC \parallel AB \parallel EF$,

and $BC \parallel FG \parallel EH \parallel AD \parallel BC$, VI, prop. XXIV

and $\therefore$ all faces are ▱s. § 97, def. ▱

2. $\therefore AB = EF = HG = DC$,

and $BC = FG = EH = AD$. I, prop. XXIV

3. And $\because \angle FEH = \angle BAD$, VI, prop. V

$\therefore \square AC \cong \square EG$, which proves (1). I, prop. XXVI

Similarly for other opposite faces.

4. And $\because PQ \parallel SR$, and $PS \parallel QR$, VI, prop. XXIV

$\therefore PR$ is a $\square$, which proves (2). § 97, def. $\square$

COROLLARY. *A parallelepiped has three sets of parallel and equal edges, four in each set.*

388. Definition. If the faces of a parallelepiped are all rectangles, it is called a **rectangular parallelepiped.**

It will be noticed that as axes of symmetry enter into the study of plane figures (§ 68), and especially of regular figures, so planes of symmetry and axes of symmetry enter into the study of solids. A *plane of symmetry* divides the solid into halves, related to each other as a figure is related to its image in a mirror. Planes of symmetry play an important part in the study of crystals. The term *axis of symmetry* will be understood from Plane Geometry.

Exercises. 618. Prepare a table showing the number (1) of faces, (2) of edges, (3) of vertices, (4) of sides in each face, (5) of plane angles at each vertex, of all of the five regular polyhedra.

619. How many degrees in the sum of the face-angles at one vertex of a regular tetrahedron? hexahedron? octahedron? dodecahedron? icosahedron?

620. The perpendiculars to the faces, through their centers, of a regular tetrahedron are concurrent in a point equidistant from all of the vertices, from all of the faces, and from all of the edges.

621. Prove that no polyhedron can have less than six edges.

622. In a regular tetrahedron three times the square on an altitude equals twice the square on an edge.

623. Certain crystals have their corners cut off, that is, the vertices of their polyhedral angles replaced by planes. Suppose a regular hexahedral crystal has its trihedral angles replaced by planes, how many faces has the new crystal? How many edges? vertices? Is Euler's theorem satisfied?

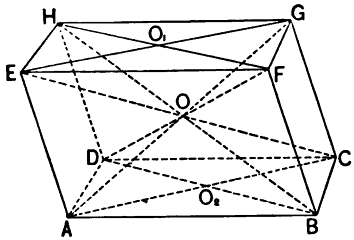
624. How many planes of symmetry and how many axes of symmetry has a regular hexahedron? octahedron?

Proposition IV.

389. Theorem. *In any parallelepiped,*

1. The four diagonals are concurrent in the mid-point of each.

2. The sum of the squares on the four diagonals equals the sum of the squares on the twelve edges.



Given a parallelepiped with diagonals AG, BH, CE, DF.

To prove that (1) the diagonals are concurrent at O, the mid-point of each;

(2) $AG^2 + BH^2 + CE^2 + DF^2 = AB^2 + BC^2 + \cdots$

Proof. 1. $\because BF =$ and $\parallel DH$, Why?

$\therefore DBFH$ is a $\square$. Why?

2. $\therefore FD$ and BH bisect each other at O. § 100, cor. 2
Similarly, BH and CE, CE and AG, bisect each other.

3. And $\because$ there is only one point of bisection of BH and CE, § 41

$\therefore BH$, CE, AG, and DF are concurrent at O.

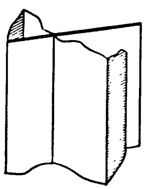
4. And $\because AG^2 + CE^2 = AC^2 + CG^2 + GE^2 + EA^2$,
and $DF^2 + BH^2 = BF^2 + FH^2 + HD^2 + DB^2$,
II, prop. XI, cor.

5. $\therefore$ by adding, and noting that $AC^2 + DB^2 = AB^2 + BC^2 + CD^2 + DA^2$, etc., the theorem is proved.

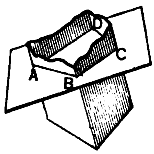
3. PRISMATIC AND PYRAMIDAL SPACE.
PRISMS AND PYRAMIDS.

390. Definitions. A **prismatic surface** is a surface made up of portions of planes, the intersections of which are all parallel to one another.

391. If, counting from any plane of a prismatic surface as the first, each plane intersects its succeeding plane, and the last one intersects the first, the surface is said to enclose a **prismatic space.**



A prismatic surface.



A portion of a prismatic space, quadrangular and convex. *ABCD*, a right section.

The lines of intersection are called the **edges,** and the portions of the planes between the edges, the **faces,** of the prismatic space.

The edges and the faces are supposed to be unlimited in length. It will be readily seen that a prismatic space is related to entire space as a plane polygon is to its entire plane. It will therefore be inferred that theorems relating to polygons have corresponding theorems relating to prismatic spaces.

392. A section of a prismatic space, made by a plane cutting its edges, is called a **transverse section.** If it is perpendicular to the edges, it is called a **right section.**

393. A prismatic space is said to be **triangular, quadrangular, rectangular, pentagonal, ..., n-gonal,** according as a transverse section is a *triangle, quadrilateral, rectangle, pentagon, ...*, *n-gon,* and to be *convex* or *concave* according as a transverse section is a convex or a concave polygon.

Prismatic spaces may be such that transverse sections are convex, concave, or cross polygons. All theorems not involving mensuration will at once be seen to apply to each class. But on account of the complexity of the figures, the third form (cross) is not considered in this work.

394. The portion of a prismatic space included between two parallel transverse sections is called a **prism,** the two transverse sections being called the **bases** of the prism.

Thus in the figure on p. 293 the portion of the prismatic space P, between S and S', is a prism. S and S' are the bases.

The signification of the terms *edges*, *faces*, and *prismatic surface* of a *prism*, *upper* and *lower bases* of a *prism*, *triangular prisms*, etc., will be inferred from the above definitions. By transverse and right sections of a prism are to be understood the transverse and right sections of its prismatic space.

The sides of the *bases* of a *prism* are also called edges; where confusion is liable to arise these are called *base edges*, and the edges of the prismatic space are called *lateral edges*.

Exercises. 625. In the figure of prop. IV, prove that O_1, O, O_2 are collinear.

626. Also that $O_1O = EA/2$.

627. Also that if AG is a rectangular parallelepiped, O_1O is perpendicular to line EG.

628. Also that if the diagonals of all the faces are drawn, and the points of intersection of the diagonals of the opposite faces are connected, these connecting lines are concurrent at O, the mid-point of each.

629. Prove that the square on a diagonal of a rectangular parallelepiped equals the sum of the squares on three concurrent edges.

630. If the edge of a cube is represented by $\sqrt{3}$, find the diagonal.

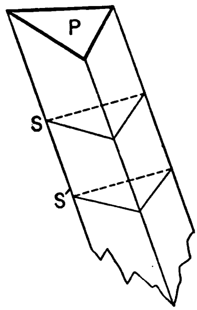
631. Prove that the four diagonals of a rectangular parallelepiped are equal.

632. Show that the edge, diagonal of a face, and diagonal, of a cube, are proportional to 1, $\sqrt{2}$, $\sqrt{3}$.

633. If the plane PR, of prop. III, were also to cut the faces HF and DB, what would be the plane figure resulting? What would be the relation of its opposite sides?

PROPOSITION V.

395. Theorem. *Parallel transverse sections of a prismatic space are congruent polygons.*



Given the prismatic space P, with S, S', two parallel transverse sections.

To prove that $S \cong S'$.

Proof. 1. $\because$ the sides of $S \parallel$ sides of S', respectively, VI, prop. XXIV

$\therefore$ $\angle$s of $S = \angle$s of S', respectively. VI, prop. V

2. And $\because$ the sides of S also equal the sides of S', respectively, I, prop. XXIV

$\therefore$ by superposition, S is evidently congruent to S'.

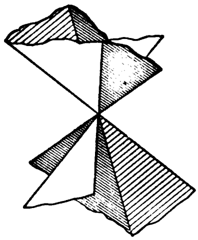
COROLLARIES. 1. *The bases of a prism are congruent polygons.*

2. *The faces of a prism are parallelograms.*

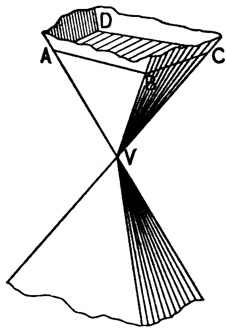
3. *The lateral edges of a prism are equal.*

Exercise. 634. Suppose in the figure of prop. III, another plane $\parallel$ to PR, cutting the same faces as PR, but not the other faces. Prove that it would cut out a parallelogram congruent to PR.

396. Definitions. A **pyramidal surface** is a surface made up of portions of planes which have but one point in common.



A pyramidal surface.



A portion of a pyramidal space, quadrangular and convex. V, the vertex; $ABCD$, a transverse section; $V\text{-}ABCD$, a pyramid, $ABCD$ being its base.

397. If, counting from any plane of a pyramidal surface as the first, each plane intersects its succeeding plane, and the last one intersects the first, the surface is said to contain a **pyramidal space.**

Unlike a prismatic space, a pyramidal space is double, its parts lying on opposite sides of the common point.

The lines of intersection of the planes are called the **edges,** the portions of the planes between the edges, the **faces,** and the point of intersection of the edges, the **vertex,** of the pyramidal space.

The edges and faces are supposed to be unlimited in length.

398. A section of a pyramidal space, made by a plane cutting all of its edges on the same side of the vertex, is called a **transverse section.**

399. The terms **triangular, ..., n-gonal, concave, convex** pyramidal space are defined as the like terms for prismatic space.

400. The portion of a pyramidal space included between the vertex and a transverse section is called a **pyramid,** the transverse section being called its **base,** and the vertex of the space, the **vertex** of the pyramid.

Thus the figure $V\text{-}ABC$ below is a pyramid, ABC being the base and V the vertex.

The distance from the vertex of a pyramid to the plane of its base is called the **altitude** of the pyramid.

Thus in the figure below, VV' is the altitude of pyramid $V\text{-}ABC$.

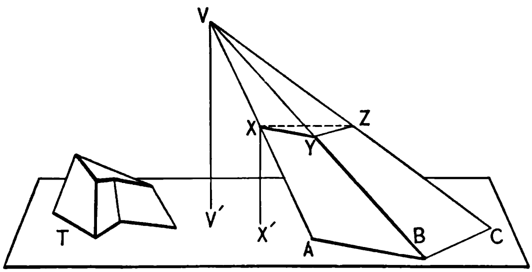
The signification of the terms *edges, faces, transverse section, base edges,* etc., of a pyramid can be inferred from the preceding definitions.

401. The portion of a pyramidal space included between two transverse sections on the same side of the vertex is called a **truncated pyramid**; if the transverse sections are parallel, it is called a **frustum** of a pyramid, the two sections being called the **bases** of the frustum.

A frustum of a pyramid is therefore a special form of a truncated pyramid.

A pyramid is also a special case of a truncated pyramid, the upper base being zero.

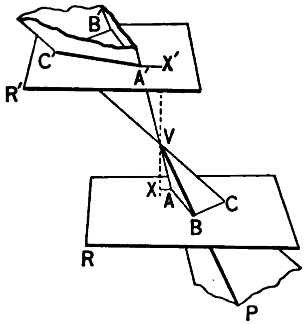
The distance from any point in one base of a frustum of a pyramid to the plane of the other base is called the **altitude** of the frustum.



T, a truncated pyramid; $ABCXYZ$, a frustum of the pyramid $V\text{-}ABC$; VV', the altitude of the pyramid; ABC, XYZ, the lower and upper bases of the frustum; XX, the altitude of the frustum.

Proposition VI.

402. Theorem. ***Parallel transverse sections of a pyramidal space are similar polygons, whose areas are proportional to the squares of the distances from the vertex to the cutting planes.***



Given P, a pyramidal space with vertex V, cut by two parallel planes, R, R', making transverse sections $ABC \cdots\cdots = S$, $A'B'C' \cdots\cdots = S'$, respectively; $VX \perp R$, $VX' \perp R'$.

To prove that (1) $S \backsim S'$,

(2) $S : S' = VX^2 : VX'^2$.

Proof. 1. $\because$ the sides of S are $\parallel$ to the sides of S', VI, prop. XXIV

$\therefore$ $VA : VA' = VB : VB' = \cdots\cdots$ and so on for other points. IV, prop. X, cor. 2

2. $\therefore$ $S \backsim S'$, which proves (1). § 258, def. sim. figs.

3. And $\because$ $AB : A'B' = VA : VA' = VX : VX'$, IV, prop. X, cors. 1, 3

and $S : S' = AB^2 : A'B'^2$, V, prop. IV

$\therefore$ $S : S' = VX^2 : VX'^2$. IV, prop. VII

NOTE. The definition of similar figures, given in Book IV, § 258, is general; the center of similitude and the given figures may or may not be in the same plane. Thus in the figure on p. 296, V is the center of similitude of the triangles ABC and $A'B'C'$, and in the figure on p. 295, V is the center of similitude of the triangles XYZ and ABC. Neither is the definition limited to plane figures; we may have similar solids as well. Thus two balls are similar, or two cubes, or two regular tetrahedra, etc.

COROLLARIES. 1. *If a pyramid is cut by a plane parallel to the base,* (1) *the edges and altitude are divided proportionally,* (2) *the section is similar to the base.*

If the planes in the proof on p. 296 are on the same side of V, step 3 proves (1), and step 2 proves (2). Or, in the figure on p. 295,

$$VV' : XX' = VA : XA,$$

and

$$\triangle ABC \sim \triangle XYZ.$$

2. *In pyramids having equal bases and equal altitudes, transverse sections parallel to the bases, and equidistant from them, are equal; if the bases are congruent, so are the sections.*

1. Let s, s' be the areas of the sections, b, b' the areas of the bases, d the distance of the section from the vertex, and a the altitude.

2. Then from prop. VI,

$$s : b = d^2 : a^2,$$

and

$$s' : b' = d^2 : a^2.$$

$$\therefore s : b = s' : b'. \qquad \text{Ax. 1}$$

3. But $b = b'$,

$$\therefore s = s'.$$

4. And if the bases are congruent, so are the sections, since they are both similar and equal to the bases.

3. *The bases of a frustum of a pyramid are similar figures.*

For they are parallel transverse sections of a pyramidal space. Hence step 2, p. 296, proves the corollary.

Exercise. 635. In the figure on p. 296, suppose $\angle ABC$ a right angle, $AB = 3$ in., $AC = 5$ in., $VB = 10$ in., and the area of $\triangle A'B'C' = 12$ sq. in.; find the length of VB'.

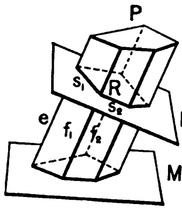
4. THE MENSURATION OF THE PRISM.

403. Definition. The area of the prismatic surface of a prism is called the **lateral area** of the prism.

Similarly for the pyramid, and for the cylinder and cone, to be defined hereafter.

PROPOSITION VII.

404. Theorem. *The lateral area of a prism equals the product of an edge and the perimeter of a right section.*



Given the prism P; a right section R with sides s_1, s_2, ... ; f_1, f_2, ... the faces of the prism; e, an edge.

To prove that the lateral area of P is $e \cdot (s_1 + s_2 + \cdots\cdots)$.

Proof. 1. $\because$ by definition of right section, $R \perp e$, $\therefore s_1 \perp e$. § 339

2. $\because f_1, f_2, \ldots\ldots$ are ▱s, Prop. V, cor. 2

$\therefore$ area $f_1 = e \cdot s_1$. V, prop. II, cor. 3

3. And $\because$ the edges are equal, Prop. V, cor. 3

$\therefore$ area $f_2 = e \cdot s_2$, and so for the other faces.

4. $\therefore$ lateral area $= e \cdot s_1 + e \cdot s_2 + \cdots\cdots = e \cdot (s_1 + s_2 + \cdots\cdots)$. Ax. 2

405. Definitions. A prism whose edges are perpendicular to the base is called a **right prism**; if the edges are oblique to the base it is called an **oblique prism.**

E.g. on p. 301 R is a right prism and O is an oblique prism. So a cube is a special kind of a right prism, and the parallelepiped illustrated on p. 290 is an oblique prism.

The distance from any point in one base of a prism to the plane of the other base is called the **altitude** of the prism.

Similarly for a parallelepiped, which is a special kind of prism.

COROLLARY. *The lateral area of a right prism equals the product of the altitude and the perimeter of the base.*

For the altitude here equals the edge.

Exercises. 636. Required the lateral area of a prism of edge 3 in., the right section being an equilateral triangle of area $\frac{1}{4}\sqrt{3}$ sq. in. Also the lateral area of one of edge 3 in., the right section being a square of diagonal $\sqrt{2}$ in.

637. Required the lateral area of a right prism whose base is a square of area 9 sq. in., and whose altitude equals the diagonal of the base. Also required the total area.

638. Required the total area of a right prism whose base is an equilateral triangle of area $\frac{1}{4}\sqrt{3}$, and whose altitude equals a base edge.

639. Required the total area of a right prism whose base is a regular hexagon whose side is 1 in., the altitude of the prism being equal to the diameter of the circle circumscribing the base.

640. Required the lateral area of a prism of edge $\frac{1}{6}$, the right section being a regular hexagon of area $\frac{3}{2}\sqrt{3}$.

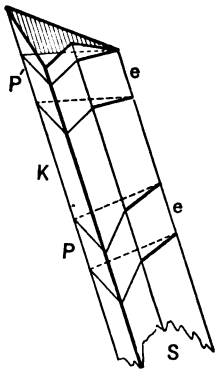
641. Required the total area of the prism mentioned in ex. 640, supposing it to be a right prism.

642. A converse of prop. VI is as follows: If two similar polygons have their corresponding sides parallel, and lie in different planes, the lines through their corresponding vertices are concurrent. Prove it. (A generalization of the idea of similar figures in perspective; see the definition of similar figures § 258, and the note at the top of p. 297.)

643. Investigate and prove whether or not any three faces of a tetrahedron are together greater than the fourth.

Proposition VIII.

406. Theorem. *Prisms cut from the same prismatic space and having equal edges are equal.*



Given two prisms, P, P', cut from the same prismatic space S, and having equal edges e.

To prove that $P = P'$.

Proof. 1. If $K =$ the portion of the prismatic space between P and P', then by adding e to the edges of K, each edge of $P + K =$ an edge of $K + P'$. Ax. 2

2. Then $\because$ $P + K$ can evidently slide along in the prismatic space and occupy the position of $K + P'$,

$\therefore P + K \cong K + P'$. § 57

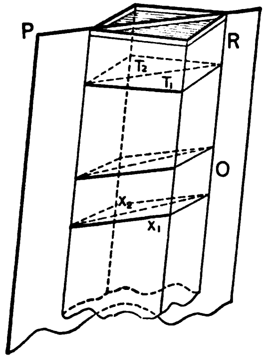
3. $\therefore P = P'$. Ax. 3

Corollaries. 1. *Right prisms having equal altitudes and congruent bases are congruent.*

2. *An oblique prism is equal to a right prism whose base and altitude are respectively a right section and edge of the oblique prism.*

PROPOSITION IX.

407. Theorem. *The two triangular prisms into which any parallelepiped is divided by a plane through two opposite edges are equal.*



Given O and R, parallelepipeds with equal edges, cut from a prismatic space, R being right; also, P, a plane through two opposite edges of that space, cutting R, O into two triangular prisms, T_1 and T_2, X_1 and X_2, respectively.

To prove that (1) $T_1 = T_2$, (2) $X_1 = X_2$.

Proof. 1. The base of $T_1 \cong$ the base of T_2. I, prop. XXIV

2. $\because$ they have the same altitude, $\therefore T_1 = T_2$. Why?

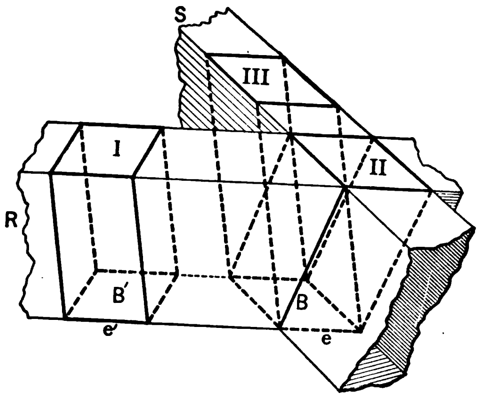
3. $X_1 = T_1$, and $X_2 = T_2$. Prop. VIII

4. And $\because T_1 = T_2$, $\therefore X_1 = X_2$. Ax. 1

COROLLARY. *A triangular prism is half of a parallelepiped of the same altitude, whose base is the parallelogram of which one side of the triangular base is the diagonal and the other two are the sides.* (Why?)

Proposition X.

408. Theorem. *Any parallelepiped is equal to a rectangular parallelepiped of equal base and equal altitude.*



Given a parallelepiped, III.

To prove that III equals a rectangular parallelepiped as I, of equal base and equal altitude.

Proof. 1. Let II be a parallelepiped on the same base, B, as III, formed by a rectangular prismatic space, R, cutting the prismatic space S of the figure.

Let I be a rectangular parallelepiped cut from R, with a base $B' = B$, and a base edge e' consequently equal to base edge e of II. II, prop. I, cor. 4

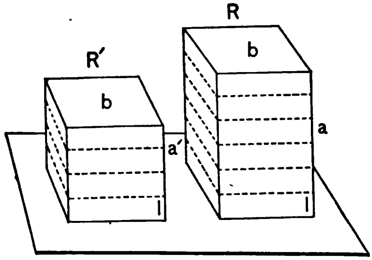
2. Then III = II, being part of S and having a common edge. Prop. VIII

3. And I = II, being part of R and having an equal edge. Prop. VIII

4. $\therefore$ III = I. Ax. 1

PROPOSITION XI.

409. Theorem. *Two rectangular parallelepipeds having congruent bases are proportional to their altitudes.*



Given two rectangular parallelepipeds, R and R', with altitudes a and a' respectively, and with bases b.

To prove that $R : R' = a : a'$.

Proof. 1. Suppose a and a' divided into equal segments, l, and suppose $a = nl$, and $a' = n'l$.

(In the figures, $n = 6$, $n' = 4$.)

Then if planes pass through the points of division, parallel to the bases,

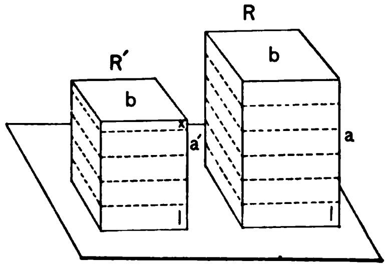
$R = n$ congruent rectangular ppds. bl,
and $R' = n'$ " " " "

2. $\therefore \dfrac{R}{R'} = \dfrac{n \cdot bl}{n' \cdot bl} = \dfrac{n}{n'} = \dfrac{a}{a'}$. Why?

NOTE. The student should notice the resemblance between this theorem and Bk. V, prop. X. The above proof assumes that a and a' are commensurable, and hence that they can be divided into equal segments l. The proposition is, however, entirely general. The proof on p. 304 is valid if a and a' are incommensurable.

Exercise. 644. Given the diagonals, a, b, c, of three unequal faces of a rectangular parallelepiped, to compute the edges.

410. Proof for incommensurable case.



1. Suppose a divided into equal segments l,

   and suppose $a = nl$, while $a' = n'l$ + some remainder x, such that $x < l$.

   Then if planes pass through the points of division, parallel to the bases,

   $R = n$ congruent rectangular ppds. bl,
   and $R' = n'$ " " " " + a remainder bx such that $bx < bl$.

2. Then a' lies between $n'l$ and $(n'+1)\,l$, Why?
   and R' lies between $n' \cdot bl$ and $(n'+1)\,bl$. Why?

3. $\therefore \frac{a'}{a}$ lies between $\frac{n'}{n}$ and $\frac{n'+1}{n}$,

   while $\frac{R'}{R}$ lies between $\frac{n'}{n}$ and $\frac{n'+1}{n}$.

4. $\therefore \frac{a'}{a}$ and $\frac{R'}{R}$ differ by less than $\frac{1}{n}$. Why?

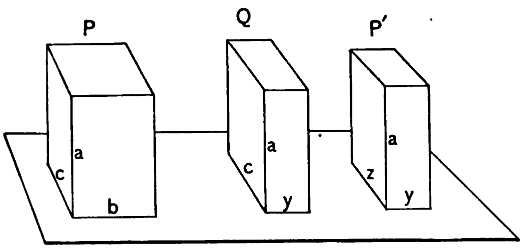
5. And $\because \frac{1}{n}$ can be made smaller than any assumed difference, by increasing n,

   $\therefore$ to assume any difference leads to an absurdity.

6. $\therefore \frac{a'}{a} = \frac{R'}{R}$, whence $\frac{R}{R'} = \frac{a}{a'}$.

PROPOSITION XII.

411. Theorem. *Two rectangular parallelepipeds of equal altitudes are proportional to their bases.*



Given two rectangular parallelepipeds, P, P', having altitudes a, and bases bc and yz, respectively.

To prove that $P : P' = bc : yz$.

Proof. 1. Suppose a rectangular parallelepiped Q to have an altitude a and a base yc.

2. Then $\because$ ac may be considered the base of P and Q,

$$\therefore \frac{P}{Q} = \frac{b}{y}.$$ Prop. XI

3. And similarly, $\frac{Q}{P'} = \frac{c}{z}$. Why?

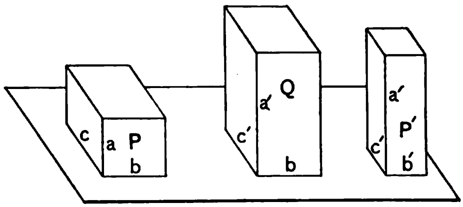
4. $$\therefore \frac{P}{P'} = \frac{bc}{yz}.$$ Why?

412. Definition. The length, breadth, and thickness of a rectangular parallelepiped are called its **three dimensions.**

Exercise. 645. If through a point on a diagonal plane of a parallelepiped planes are passed parallel to the two pairs of faces not intersected by the diagonal, the parallelepipeds on opposite sides of that diagonal plane are equal. (See II, prop. IV.)

Proposition XIII.

413. Theorem. *Two rectangular parallelepipeds are proportional to the products of their three dimensions.*



Given two rectangular parallelepipeds, P, P', of dimensions a, b, c, and a', b', c', respectively.

To prove that $P : P' = abc : a'b'c'$.

Proof. 1. Suppose a rectangular parallelepiped Q to have the three dimensions a', b, c'.

2. Then $\dfrac{P}{Q} = \dfrac{ac}{a'c'}$, Prop. XII

and $\dfrac{Q}{P'} = \dfrac{b}{b'}$. Prop. XI

3. $\therefore \dfrac{P}{P'} = \dfrac{abc}{a'b'c'}$. Why?

Corollaries. 1. *The volume of a rectangular parallelepiped equals the product of its three dimensions.*

This means that the *number* which represents the volume is the product of the three *numbers* representing the dimensions. That is, the *number* of times the unit of volume is contained in the given parallelepiped, is the product of the *numbers* of times the unit of length is contained in three concurrent edges.

If P' is a cube, of edges 1, 1, 1; then P' is the unit of measure of volume. But $P : P'$ then becomes $P : 1$, and $abc : 1 \cdot 1 \cdot 1$ then becomes $abc : 1$. $\therefore P : 1 = abc : 1$, or $P = abc$.

2. *The volume of any parallelepiped equals the product of its base and altitude.*

For (prop. X) it equals a rectangular parallelepiped of equal base and equal altitude.

3. *The volume of a triangular prism equals the product of its base and altitude.*

Cor. 2 with prop. IX, cor. Let the student give the proof in detail.

4. *The volume of any prism equals the product of its base and altitude.*

For it can be cut into triangular prisms by diagonal planes through a lateral edge, the sum of the bases of the triangular prisms being the base of the given prism. $\therefore$ cor. 3 applies. Let the student draw the figure and give the proof in detail.

5. *Any prism equals a rectangular parallelepiped of equal base and equal altitude.*

Cors. 4, 2.

6. *The volume of an oblique prism equals the product of an edge and a right section.*

Cor. 4 with prop. VIII, cor. 2.

7. *Prisms having equal bases are proportional to their altitudes.*

For if a is the altitude and b the base, then $P = ab$, and $P' = a'b'$. If $b = b'$, then $P' = a'b$. Hence $P : P' = ab : a'b = a : a'$.

8. *Prisms having equal altitudes are proportional to their bases. Prisms having equal bases and equal altitudes are equal.*

Let the student give the proof.

Exercises. **646.** What is the edge of the cube whose volume equals that of a rectangular parallelepiped with edges 2.4 m, 0.9 m, 0.8 m ?

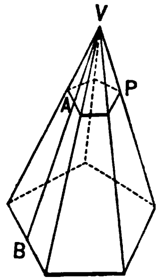
647. From the given edge e of a cube, compute (1) the cube's entire surface, (2) its diagonal, (3) its volume.

648. Draw a figure illustrating geometrically the formula

$$(a + b)^3 = a^3 + b^3 + 3\,a^2b + 3\,ab^2.$$

5. MENSURATION OF THE PYRAMID.

414. Definitions. A **regular pyramid** is a pyramid whose base is a regular convex polygon, the perpendicular to which, at its center, passes through the vertex of the pyramid.



415. The **slant height** of a *regular* pyramid is the distance from the vertex to any side of the base.

E.g. VB in the annexed figure.

416. The portion of the slant height of a *regular* pyramid cut off by the bases of a frustum is called the **slant height** of the **frustum.**

COROLLARY. *The slant height of a regular pyramid, or of a frustum of a regular pyramid, is the same on whatever face it is measured.*

Let the student show that the faces are all congruent; hence that the slant heights are equal.

Exercises. 649. To pass a plane through a given pyramid parallel to the base, so that the section shall equal half the base.

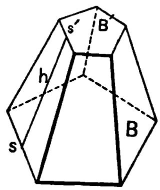
650. The edges of a rectangular parallelepiped are 3, 4, 5; required the total area of the faces, the areas of its diagonal planes, the length of its diagonal line, and the lengths of the diagonals of its faces. Similarly for a cube of edge $\sqrt{2}$.

651. If a cubic block of sandstone at a temperature of 0° Centigrade has an edge 1 m long, and if for every 1° Centigrade increase of temperature the edge increases 0.000012 of its length at 0°, find the volume at 40° Centigrade.

652. A brick has the dimensions 25 cm, 12 cm, 6 cm, but on account of shrinkage in baking, the mold is 27.5 cm long, and proportionally wide and deep. What per cent does the volume of the brick decrease in baking?

PROPOSITION XIV.

417. Theorem. *The lateral area of the frustum of a regular pyramid equals half the product of its slant height and the sum of the perimeters of its bases.*



Given BB', a frustum of a regular pyramid, h its slant height, s a side of base B and p its perimeter, s' a side of base B' and p' its perimeter, l the lateral area.

To prove that $l = \frac{1}{2} h (p + p')$.

Proof. 1. The area of each face $= \frac{1}{2} h (s + s')$.

V, prop. II, cor. 5

2. Adding all the faces, and remembering that p is the sum of the sides s, and p' of the sides s', we have $l = \frac{1}{2} h (p + p')$.

COROLLARY. *The lateral area of a regular pyramid equals half the product of its slant height and the perimeter of its base.*

For in the above theorem, let $B' = 0$; then s' and $p' = 0$; $\therefore l = \frac{1}{2} hp$.

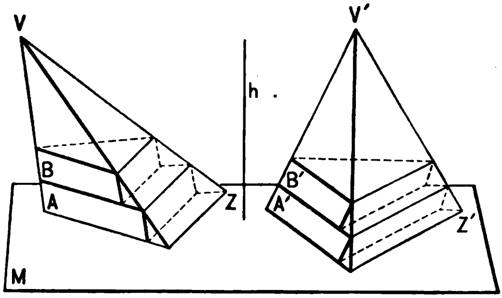
Exercises. 653. Prove the above corollary independently of the theorem.

654. What is the lateral area of a regular pyramid whose base is a triangle of altitude $\frac{a}{2}\sqrt{3}$, and whose slant height is a?

655. What is the total area of a frustum of a regular hexagonal pyramid whose base edges are respectively $3 - \sqrt{3}$ and $3 + \sqrt{3}$, and whose slant height is 10?

Proposition XV.

418. Theorem. *Pyramids having equal bases and equal altitudes are equal.*



Given pyramids VAZ, $V'A'Z'$, having equal bases, and having equal altitudes h.

To prove that pyramid $VAZ =$ pyramid $V'A'Z'$.

Proof. 1. Suppose their bases in the same plane M, and their vertices on the same side of M.

Suppose their altitude h divided into n equal parts and planes passed through the division-points parallel to M.

Then these planes will make equal corresponding transverse sections because the bases are equal.

Prop. VI, cor. 2

2. Suppose planes passed through the sides of these sections parallel to an edge of the pyramid, making a set of prisms in each pyramid, A, B, ... and A', B', ...

3. Then $\because A = A'$,

and $B = B'$, ... Prop. XIII, cor. 8

$\therefore A + B + \cdots\cdots = A' + B' + \cdots\cdots$ Ax. 2

4. But if n increases indefinitely,

$$A + B + \cdots\cdots \doteq \text{pyr. } VAZ,$$

and $$A' + B' + \cdots\cdots \doteq \text{pyr. } V'A'Z'.$$

5. $\therefore$ pyr. $VAZ =$ pyr. $V'A'Z'$. IV, prop. IX, cor. 1

COROLLARIES. 1. *A pyramid having a parallelogram for its base is divided into equal pyramids by a plane through its vertex and two opposite vertices of the base.*

For the two pyramids have equal bases and a common altitude.

2. *A pyramid having a parallelogram for its base equals twice a triangular pyramid of the same altitude, whose base equals half that parallelogram.* (Why ?)

3. *A triangular pyramid can be constructed equal to any given n-gonal pyramid.*

II, prop. XII, and this theorem.

Exercises. 656. Find the area of the entire surface of a regular tetrahedron of altitude h.

657. Find the altitude of a regular tetrahedron of total area a.

658. Find, by § 417, the total area of a cube of edge e.

659. What is the length of the base edge of a regular triangular pyramid which is equal to a regular hexagonal pyramid of the same altitude, the base edge being 1 ?

660. In prop. XIV, cor., B' was supposed to decrease to 0 ; supposing, instead, that B' increases until it equals B, show that step 2 of the theorem gives the usual formula for the lateral area of a prism.

661. Prove that frustums of pyramids having equal bases and equal altitudes, which themselves have equal altitudes, are equal.

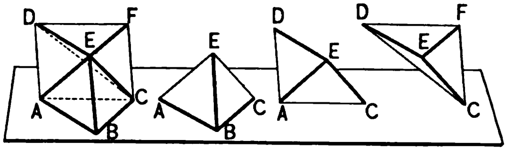
662. A pyramid has for its base a regular hexagon with its shorter diagonal $\sqrt{3}$; the altitude equals the longer diagonal ; required the lateral area of the pyramid.

663. Find the total area of the pyramid mentioned in ex. 662.

664. The lower base of a frustum of a regular pyramid is a square of area s^2 ; the area of the upper base is half that of the lower one ; the slant height is s ; required the lateral area.

Proposition XVI.

419. Theorem. *A triangular prism can be divided into three equal triangular pyramids.*



Given $ABCDEF$, a triangular prism.

To prove that $ABCDEF$ can be divided into three equal triangular pyramids.

Proof. 1. $\because A, E, C$ determine a plane, also C, D, E, § 330, 1
$\therefore ABCDEF =$ three triangular pyramids, viz., $E\text{-}ABC$, $E\text{-}ACD$, $C\text{-}DEF$. Ax. 8

2. But $\because \triangle ABC \cong \triangle DEF$,
$\therefore E\text{-}ABC = C\text{-}DEF$. Prop. XV

3. And $C\text{-}DEF \equiv E\text{-}DCF = E\text{-}ACD$, because they have a common altitude from E to plane $ACFD$, and equal bases. Prop. XV

4. $\therefore E\text{-}ABC = C\text{-}DEF = E\text{-}ACD$. Ax. 1

Corollaries. 1. *A triangular pyramid is one-third of a triangular prism of the same base and same altitude.*

For the prism is three times the pyramid.

2. *Any pyramid is one-third of a prism of the same base and same altitude.*

For, dividing the base into triangles by drawing diagonals, the pyramid may be considered as made up of triangular pyramids, each of which is a third of a triangular prism of the same base and same altitude; hence the sum of the triangular pyramids, or the given pyramid, equals one-third the sum of the triangular prisms, or one-third of a prism of the same base and same altitude.

3. *The volume of a pyramid equals one-third the product of its base and altitude.*

Cor. 2, and prop. XIII, cor. 4.

4. *Pyramids having equal bases are proportional to their altitudes; having equal altitudes, to their bases.*

For if $p = \frac{1}{3}ab$, and $p' = \frac{1}{3}a'b'$, then $\dfrac{p}{p'} = \dfrac{\frac{1}{3}ab}{\frac{1}{3}a'b'} = \dfrac{ab}{a'b'}$.

And if $b = b'$, then $\dfrac{ab}{a'b'} = \dfrac{a}{a'}$.

Or if $a = a'$, then $\dfrac{ab}{a'b'} = \dfrac{b}{b'}$.

Or if $a = a'$ and $b = b'$, then $\dfrac{p}{p'} = 1$, or $p = p'$, as stated in prop. XV.

420. Definitions. A polyhedron which has for bases any two polygons in parallel planes, and for lateral faces triangles or trapezoids which have one side in common with one base and the opposite vertex or side in common with the other base, is called a **prismatoid.**

The **altitude** of a prismatoid is the perpendicular distance between the planes of its bases.

Exercises. 665. Find the volume of the pyramid mentioned in ex. 662.

666. A church-tower is capped by a regular octagonal pyramid whose height is 55.5 m, and whose base edge is 4.9 m. Required the volume.

667. A pentagonal pyramid has equal lateral and base edges, 1 in. Find the lateral area.

668. Find the volume of a cube the diagonal of whose face is $a\sqrt{2}$.

669. Each face of a given triangular pyramid is an equilateral triangle whose side is 3. Find the total area.

670. Find the volume of the tetrahedron mentioned in ex. 656.

671. Also of the pyramid mentioned in ex. 667.

672. An edge of a regular octahedron is 1 in. Find the volume.

673. A pyramid stands on a square base of edge 1 m; the lateral edge of the pyramid is also 1 m. Find the lateral area and volume.

PROPOSITION XVII.

421. Theorem. *The volume of a prismatoid of bases* b *and* b′, *altitude* h, *and transverse section* m *midway between the bases, is expressed by the formula*

$$v = \frac{h}{6}(b + b' + 4m).$$

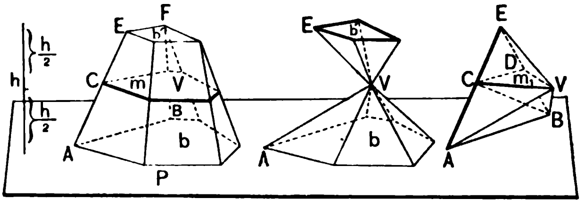


FIG. 1. FIG. 2. FIG. 3.

Proof. 1. If any face, *ABFE*, of the prismatoid *P*, is a trapezoid, divide it into two triangles by a diagonal *EB*. Let *V* be any point in *m*; join *V* to the vertices of *P*; then *P* will be divided into two pyramids (Fig. 2) of bases *b*, *b′* and vertex *V*, and also pyramids of vertex *V* and triangular bases *ABE*, etc. (Fig. 3.) Let *EB* meet *m* at *D*; call $\triangle\ VDC\ m_1$. (Fig. 3.)

2. Then the volume of

$$V\text{-}b = \tfrac{1}{3}\,b \cdot \frac{h}{2},$$

and $$V\text{-}b' = \tfrac{1}{3}\,b' \cdot \frac{h}{2}.$$ Prop. XVI, cor. 3

This completes Fig. 2.

3. Pyramid $V\text{-}ABE = E\text{-}CVD + B\text{-}CVD + V\text{-}ABC.$ Ax. 8

4. Of these, $E\text{-}CVD = \tfrac{1}{3}\,m_1 \cdot \frac{h}{2},$

and $B\text{-}CVD = \tfrac{1}{3}\,m_1 \cdot \frac{h}{2}.$ Prop. XVI, cor. 3

5. But $V\text{-}ABC =$ twice $V\text{-}CBD$ (or $B\text{-}CVD$),

$\because \triangle ABC =$ twice $\triangle CBD$,

having edge $AB = 2 \cdot CD$, and a common altitude.

Prop. XVI, cor. 4

6. $\therefore V\text{-}ABC = \frac{2}{3} m_1 \cdot \frac{h}{2}$.

7. $\therefore$ pyramid $V\text{-}ABE = \frac{h}{6} \cdot 4 m_1$, Axs. 2, 8

and $\therefore$ the sum of the pyramids of the form of

$V\text{-}ABE = \frac{h}{6} \cdot 4 m$. Axs. 2, 8

8. $\therefore P = \frac{h}{6}(b + b' + 4m)$. Axs. 2, 8

NOTE. The Prismatoid Formula, $v = \frac{h}{6}(b + b' + 4m)$, is of great value in the mensuration of solids. From it can be derived formulæ for the volumes of all of the solids of elementary geometry.

COROLLARY. *The volume of the frustum of a pyramid, of bases* b, b′, *and altitude* h, *is* $\frac{h}{3}(b + b' + \sqrt{bb'})$.

For if e, e' are corresponding sides of b, b', then $\frac{1}{2}(e + e')$ is the corresponding side of m. (Why ?)

$$\therefore \frac{e}{\frac{1}{2}(e+e')} = \frac{\sqrt{b}}{\sqrt{m}}, \text{ and } \frac{e'}{\frac{1}{2}(e+e')} = \frac{\sqrt{b'}}{\sqrt{m}}. \quad \text{(V, prop. IV.)}$$

$$\therefore \frac{e+e'}{\frac{1}{2}(e+e')} = \frac{\sqrt{b}+\sqrt{b'}}{\sqrt{m}}, \text{ and } \therefore 2\sqrt{m} = \sqrt{b} + \sqrt{b'}.$$

$\therefore 4m = b + b' + 2\sqrt{bb'}$, which may be substituted in the Prismatoid Formula.

Exercises. 674. By letting (1) $b' = 0$, and (2) $b' = b$, show that (1) prop. XVI, cor. 3, and (2) prop. XIII, cor. 4, follow, as special cases, from the Prismatoid Formula.

675. Calling a prismatoid whose lower base b is a rectangle of length l and width w, and whose upper base b' is a line e parallel to a base edge, and whose altitude is h, a *wedge*, find a formula for the volume of a wedge.

EXERCISES.

676. The base of a wedge is 4 by 6, the altitude is 5, and the edge e is 3. Find the volume. (See ex. 675.) Also, when $e = 0$.

677. The altitude of a pyramid is divided into five equal parts by planes parallel to the base. Find the ratios of the various frustums to one another and to the whole pyramid.

678. Two pyramids, P, P', have square bases, and are such that the altitude of P equals twice the altitude of P', but the base edge of P is half as long as the base edge of P'. Find the ratio of their volumes.

679. Find the volume of a cube whose diagonal is $\sqrt{3}$.

680. A frustum of a pyramid has for its bases squares whose sides are respectively 0.6 m, 0.5 m; the altitude of the frustum is 0.9 m. Find the volume.

681. Given the volume v, and the bases b, b', of a frustum of a pyramid, to find a formula for (1) its altitude, (2) the altitude of the whole pyramid.

682. A granite monument is in the form of a frustum of a square pyramid, surmounted by a pyramid; the sides of the bases of the frustum are 1 m and 0.8 m, and the altitude of the frustum is 1.8 m; the altitude of the pyramidal top is 0.45 m. A cubic meter of water weighs a metric ton, and granite is three times as heavy as water. Find the weight of the monument.

683. An excavation 1.5 m deep, rectangular at top and bottom, and in the form of a frustum of a pyramid, has its upper base 10 m wide and 16 m long, and the lower base 7.5 m wide. How many cubic meters of earth would it take to fill it to a depth of 0.75 m?

684. The volume of a cube is six times that of the regular octahedron formed by joining the centers of the faces of the cube.

685. Find the volume of a prismatoid of altitude 3.5 cm, the bases being rectangles whose corresponding dimensions are 3 cm by 2 cm, and 3.5 cm by 5 cm.

686. It is usual to find the volume of a pile of broken stones by taking the product of the altitude and the area of a transverse mid-section. Compare this with the Prismatoid Formula and find what relation it assumes between m and $b + b'$. Is this relation true in the case of a pyramid?

687. The volume of a pyramid equals the product of the altitude and a transverse section (parallel to the base) how far from the vertex?

BOOK VIII. — THE CYLINDER, CONE, AND SPHERE. SIMILAR SOLIDS.

1. THE CYLINDER.

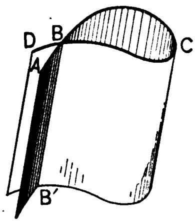
422. Definitions. A **curved surface** is a surface no part of which is plane.

The number of kinds of curved surfaces is unlimited, just as the number of kinds of curves in a plane is unlimited. But as among plane curves the circumference is the best known, so there are certain curved surfaces which are better known than others, and these are treated in this book.

423. A **cylindrical surface** is a surface generated by a straight line, called the **generatrix,** which moves so as constantly to pass through a given curve, called the **directrix,** and to remain parallel to its original position.

The surface of a piece of straight pipe, or the surface of the paper in a roll, is an example.

424. A straight line in any position of the generatrix is called an **element** of the cylindrical surface.



One form of a cylindrical surface. $ABCBD$, the directrix; BB', an element; BCB', a portion of a cylindrical space.

425. If the directrix is a closed curve, the cylindrical surface incloses a space of unlimited length, called a **cylindrical space.**

426. A section of a cylindrical space, made by a plane cutting its elements, is called a **transverse section.** If it is perpendicular to the elements, it is called a **right section.**

As a transverse section of a prismatic space may be a convex, concave, or cross polygon, so a transverse section of a cylindrical space may be a curve of any shape if only its end-points meet. All theorems, if the signs are properly considered, will be seen to apply to each of the three forms of transverse section, corresponding to convex, concave, and cross polygons. The third is, however, too complex for treatment in elementary works.

427. The portion of a cylindrical space included between two parallel transverse sections is called a **cylinder.**

E.g. the portion between planes P and P' in the figure on p. 319.

The terms *bases* and *altitude* of a cylinder will be understood, without further definition, from the corresponding definitions under the prism. The student should, throughout this section, notice the relation of cylindrical spaces to prismatic spaces.

A *cylinder* is considered as having the same *directrix* as its cylindrical space, and as having for *elements* the segments of the elements of the cylindrical surface included between its bases.

A cylinder is said to be **right** or **oblique** according as its elements are perpendicular or oblique to the bases.

If the base of a cylinder is a circle, the cylinder is said to be **circular.**

428. Postulate of the Cylinder. *A cylindrical surface may be constructed with any directrix and with any original position of the generatrix.*

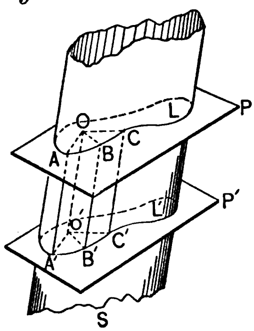
In solid geometry constructions are allowed which require other instruments than the compasses and straight-edge. For example, this postulate requires the generatrix to move constantly parallel to its original position, a construction manifestly impossible by the use of merely these two instruments.

Exercises. 688. Draw a figure of a convex cylinder; a concave cylinder; a cross cylinder.

689. Prove that if a transverse section of a cylindrical space is perpendicular to one element it is a right section.

PROPOSITION I.

429. Theorem. *Parallel transverse sections of a cylindrical space are congruent.*



Given a cylindrical space S, cut by two parallel planes, P, P', so as to form two transverse sections, L, L'.

To prove that $L \cong L'$.

Proof. 1. Let AA', BB', CC' be segments of elements between P and P', O any point in P, and $OO' \parallel AA'$, meeting P' at O'; join O to A, B, C, and O' to A', B', C'.

2. Then OO', AA' determine a plane. VI, prop. I, cor. 2

3. And $\because OA \parallel O'A'$, $OB \parallel O'B'$, $OC \parallel O'C'$, § 367
$\therefore \angle AOB = \angle A'O'B'$, $\angle AOC = \angle A'O'C'$, ... § 337

4. Also, $OA = O'A'$, $OB = O'B'$, ... I, prop. XXIV
$\therefore$ if L is placed on L' so that O falls on O' and OA lies on $O'A'$, A will fall on A', B on B', etc.

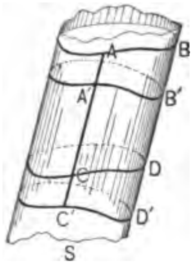
5. Similarly, for every point of L there is a single corresponding point of L' on which it will fall.
$\therefore$ the figures are congruent. § 57, def. congruence

COROLLARIES. 1. *The bases of a cylinder are congruent.*

2. *The elements of a cylinder are equal.* (Why ?)

PROPOSITION II.

430. Theorem. *Cylinders cut from the same cylindrical space, and having equal elements, are equal.*



Given two cylinders, AD, $A'D'$, cut from the same cylindrical space S, and having equal elements AC, $A'C'$.

To prove that $AD = A'D'$.

Proof. 1. $\because AC = A'C'$,

and $A'C \equiv A'C$,

$\therefore AA' = CC'$. Ax. 3

2. Similarly for BB' and DD', and for all other segments of the same elements, included between AB, $A'B'$, and CD, $C'D'$.

3. And $\because CD \cong AB$, and $C'D' \cong A'B'$, Prop. I

$\therefore$ solid CD' can be made to slide along in S and coincide with solid AB', since they are equal in all their parts.

4. Adding the common part $A'D$,

$AD = A'D'$. Ax. 2

COROLLARY. *The cylindrical surfaces of two cylinders cut from the same cylindrical space, and having equal elements, are equal.*

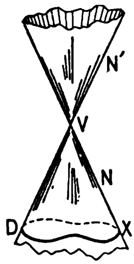
For it is proved, in step 3, that they can be made to coincide.

2. THE CONE.

431. Definitions. A **conical surface** is a surface generated by a straight line which moves so as constantly to pass through a given curve and contain a given point called the **vertex.**

The terms *generatrix*, *directrix*, *elements* will be understood from §§ 423, 424.

432. The portions of the conical surface on opposite sides of the vertex are called the **nappes,** and are usually distinguished as **upper** and **lower.**



A conical surface. DX, the directrix; V, the vertex; N, N', the lower and upper nappes; V-DX, a cone, with base the closed figure DX.

433. If the directrix is a closed curve, the conical surface incloses a double space, on opposite sides of the vertex, known as a **conical space.**

A section of a conical space made by a plane cutting all of its elements on the same side of the vertex is called a **transverse section.**

434. The portion of a conical space included between the vertex and a transverse section is called a **cone,** the transverse section being called its **base.**

A *cone* is considered as having the same *directrix* and *vertex* as its conical space, and the segments of the elements between the vertex and base are called the *elements* of the cone.

The distance from the vertex of a cone to the plane of the base is called the **altitude** of the cone.

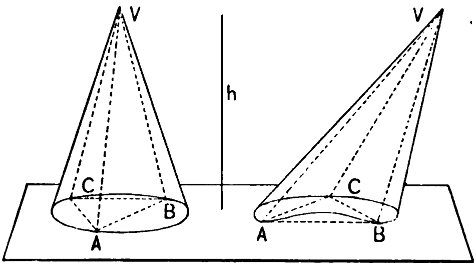
If the base of a cone is a circle, the cone is said to be **circular.** In that case the line determined by the vertex and the center of the base is called the **axis** of the cone. If this axis is perpendicular to the base, the cone is called a **right circular** cone; if oblique, an **oblique circular** cone.

A right circular cone is often called a *cone of revolution*, because it can be generated by the revolution of a right-angled triangle about one of its shorter sides. A right circular cylinder is often called a *cylinder of revolution.* (Why?)

435. Postulate of the Cone. *A conical surface may be constructed with any directrix and any vertex.*

436. Relation of Cone and Pyramid. If points A, B, C, ... are taken on the perimeter of the base of a cone, and joined to the vertex V, and if planes be passed through VA and VB, VB and VC, ..., a pyramid will be formed, called an **inscribed pyramid.**

If the base of the cone is bounded by a convex curve, the base of the pyramid will be a polygon inscribed in it. But whether the base is convex or not, the pyramid is called an *inscribed* pyramid.



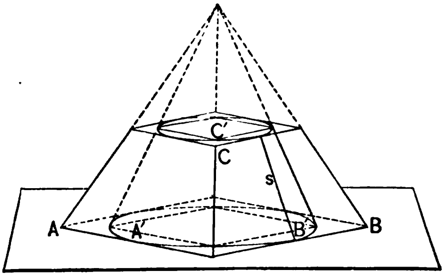
Pyramids inscribed in cones. The first figure is a right circular cone. The inscribed pyramids are indicated by dotted lines. h, the altitude.

437. If the base of the cone is a circle, and a regular polygon is circumscribed about it, the planes determined by the sides of the polygon and the vertex of the cone form, with the polygon, a pyramid which is said to be **circumscribed** about the circular cone.

There are other forms of circumscribed pyramids, but the one here mentioned is the only one that is necessary for this work.

The *slant height* of a right circular cone is defined as the slant height of the circumscribed pyramid. (Why?)

438. If a pyramid is inscribed in or circumscribed about a cone, a transverse section of the pyramid and cone cuts off, toward the base, a **frustum of a cone** and an **inscribed** or **circumscribed frustum of a pyramid.**



ABC, a circumscribed frustum of a pyramid; *A'B'C'*, an inscribed frustum of a regular pyramid; *s*, the slant height of the frustum of the cone.

The terms *bases*, *altitude*, and *lateral* surface will be understood from the terms used with the pyramid and the frustum of a pyramid.

439. From the above definitions it is evident that, if the inscribed or circumscribed frustum of a pyramid has equilateral bases, then if the number of lateral faces increases indefinitely, the frustum of the pyramid, its bases, and its lateral surface, approach as their respective limits the frustum of the cone, its bases, and its lateral surface, but that the altitude does not vary. If a frustum of a right pyramid be circumscribed about the frustum of a right circular cone, the slant height of the frustum of the pyramid may be called the *slant height of the frustum of the cone.* Hence the following

440. COROLLARY. *If* F *is the frustum of a cone, and* F' *the inscribed or circumscribed frustum of a pyramid, of equilateral bases, and if* b_1, b_2, l, v *are the bases, lateral surface, and volume, respectively, of* F, *and* b_1', b_2', l', v' *the bases, lateral surface, and volume, respectively, of* F', *then if the number of faces of* F' *increases indefinitely,*

$$b_1' \doteq b_1,\ \ b_2' \doteq b_2,\ \ l' \doteq l,\ \ v' \doteq v.$$

PROPOSITION III.

441. Theorem. *The lateral area of a frustum of a right circular cone equals one-half the product of the slant height and the sum of the circumferences of its bases.*

Given a frustum of a right circular cone, l its lateral area, c_1 and c_2 the circumferences of its upper and lower bases, respectively, and s its slant height.

To prove that $l = \frac{1}{2} s(c_1 + c_2)$.

Proof. 1. Let l', p_1, p_2, s be the lateral area, the perimeters of the upper and lower bases, and the slant height, respectively, of the circumscribed frustum F of a regular pyramid.

2. Then $l' = \frac{1}{2} s(p_1 + p_2)$. VII, prop. XIV

3. But if the number of faces of F increases indefinitely, $l' \doteq l$, $p_1 \doteq c_1$, $p_2 \doteq c_2$, while the slant height is the same. § 440

4. $\therefore l = \frac{1}{2} s(c_1 + c_2)$. IV, prop. IX, cor. 1

COROLLARIES. 1. *If the radii of the upper and lower bases are* r_1, r_2, *respectively, then* $l = \pi s(r_1 + r_2)$.

2. *If* r_3 = *the radius of the circle midway between the bases of the frustum, then* $l = 2\pi r_3 s$.

For $r_3 = (r_1 + r_2)/2$. Why?

3. *The lateral area of a right circular cone equals half the product of its slant height and the circumference of the base.*

If the upper base of a frustum of a cone decreases to zero, what does the frustum become? At the same time what does c_1 of step 4 become?

4. *The lateral area of a right circular cylinder equals the product of its altitude and the circumference of the base.*

If, in step 4, $c_1 = c_2$, what does l equal? What does s equal?

PROPOSITION IV.

442. Theorem. *The volume of the frustum of a cone of bases* b_1, b_2 *and altitude* h *is expressed by the formula*

$$v = \frac{h}{3}(b_1 + b_2 + \sqrt{b_1 b_2}).$$

Proof. 1. Let v', h, b_1', b_2' be the volume, altitude, and bases, respectively, of an inscribed frustum of a pyramid with an equilateral base.

2. Then $v' = \frac{h}{3}(b_1' + b_2' + \sqrt{b_1' b_2'})$. VII, prop. XVII

3. But if the number of faces of v' increases indefinitely, $v' \doteq v$, $b_1' \doteq b_1$, $b_2' \doteq b_2$, while h is constant. § 440

4. $\therefore v = \frac{h}{3}(b_1 + b_2 + \sqrt{b_1 b_2})$. IV, prop. IX, cor. 1

COROLLARIES. 1. *If the frustum is circular, and the radii of* b_1, b_2 *are* r_1, r_2, *respectively, then* $v = \frac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 r_2)$.

2. *If* $r_3 =$ *the radius of the circle midway between the bases of a frustum of a circular cone, and if* h *is the altitude, and* r_1, r_2 *are the radii of the bases, then* $v = \frac{1}{6}\pi h(r_1^2 + r_2^2 + 4r_3^2)$.

See prop. III, cor. 2.

3. *The volume of a cone of base* b *and altitude* h *is expressed by the formula* $v = \frac{1}{3}hb$.

Let $b_2 = 0$ in prop. IV.

4. *The volume of a circular cone, the radius of whose base is* r, *is expressed by the formula* $v = \frac{1}{3}\pi r^2 h$.

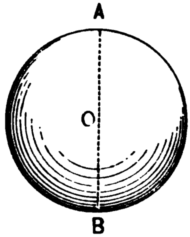
5. *The volume of a cylinder of base* b *and altitude* h *is expressed by the formula* $v = hb$.

Let $b_1 = b_2$.

6. *The volume of a cylinder of altitude* h *and base radius* r *is expressed by the formula* $v = \pi r^2 h$.

3. THE SPHERE.

443. Definitions. A **sphere** is the finite portion of space bounded by a surface, which is called a **spherical surface** and is such that all points upon it are equidistant from a point within called the **center** of the sphere.



A sphere. O, the center. OA, OB, radii. AB, a diameter.

A straight line terminated by the center and the spherical surface is called a **radius,** and a straight line through the center, terminated both ways by the spherical surface, is called a **diameter** of the sphere.

A section of a sphere made by a plane is called **a plane section.**

444. Corollaries. 1. *A diameter of a sphere is equal to the sum of two radii of that sphere.*

2. *Spheres having the same radii are congruent, and conversely.*

3. *A point is within a sphere, on its surface, or outside the sphere, according as the distance from that point to the center is less than, equal to, or greater than, the radius.*

445. Postulates of the Sphere. (Compare § 109.)

1. *All radii of the same sphere are equal, and hence all diameters of the same sphere are equal.*

2. *If an unlimited straight line passes through a point within a sphere, it must cut the surface at least twice.*

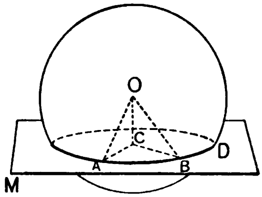
3. *If an unlimited plane, or if a spherical surface, intersects a spherical surface, it must intersect it in a closed line.*

4. *A sphere has but one center.*

5. *A sphere may be constructed with any center, and with a radius equal to any given line segment.*

PROPOSITION V.

446. Theorem. *A plane section of a sphere is a circle.*



Given a sphere with center O, and a section $ABDC$ made by a plane M.

To prove that $ABDC$ is a circle.

Proof. 1. M intersects the sphere in a closed line. § 445, 3

2. Suppose O joined to two points AB on that line, and $OC \perp M$; draw CA, CB.

3. Then $\because$ ∠s OCB, OCA are rt., and $OC \equiv OC$, and $OB = OA$,

$\therefore \triangle CBO \cong \triangle CAO$, and $CB = CA$. § 88, cor. 5

So for any other points on the closed line.

4. $\therefore ABDC$ is a circle and C is its center. § 165, def. ⊙

447. Definitions. A *great circle* of a sphere is a circle passing through its center; a *small circle*, one not passing through its center.

COROLLARIES. 1. *The line determined by the center of a sphere and the center of any small circle of that sphere is perpendicular to that circle.*

For the line OC from the center of the sphere perpendicular to the circle has been proved to coincide with the line determined by the center of the circle and the center of the sphere, and there is only one line from the center of the sphere perpendicular to the circle.

2. *Of two circles of a sphere, the first is greater than, equal to, or less than, the second, according as its distance from the center is less than, equal to, or greater than, that of the second.*

For $AC^2 = r^2 - OC^2$; $\therefore$ the smaller OC, the greater AC, etc.

3. *A great circle has the same center and radius as the sphere itself; hence all great circles of a given sphere are equal.*

4. *A great circle bisects the sphere and the spherical surface.*

For if the two parts are applied one to the other, they will coincide; if they did not, the definition of sphere would be violated.

5. *Two great circles bisect each other.*

They have the same center, and hence a common diameter.

448. The student should notice the relation between the sphere and circle. Thus in prop. V and its corollaries:

The Circle.	**The Sphere.**
A portion of a *line* cut off by a *circumference* is a *chord.*	A portion of a *plane* cut off by a *spherical surface* is a *circle.*
The greater a *chord*, the less its distance from the center.	The greater a *circle*, the less its distance from the center.
A *diameter* (great chord) bisects the *circle* and the *circumference.*	A *great circle* bisects the *sphere* and the *spherical surface.*
Two *diameters* (great chords) bisect each other.	Two *great circles* bisect each other.

Hence may be anticipated a line of theorems on the sphere, derived from those on the circle, by making the following substitutions:

1. *Circle*, 2. *circumference*, 3. *line*, 4. *chord*, 5. *diameter*.	1. *Sphere*, 2. *spherical surface*, 3. *plane*, 4. *circle*, 5. *great circle*.

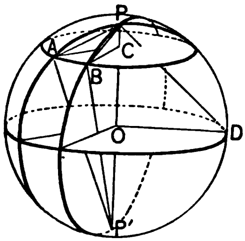
449. Definitions. The diameter of a sphere, perpendicular to a circle of that sphere, is called the **axis** of that circle, and its extremities are called the **poles** of that circle.

The two equal parts into which a great circle divides a sphere are called **hemispheres**, their curved surfaces being called **hemispherical surfaces.**

Corollary. *The axis of a circle passes through its center.*

PROPOSITION VI.

450. Theorem. *The straight lines joining any two points on the circumference of a circle of a sphere to one of the poles of that circle are equal.*



Given the circle ABC, and its poles P, P'; PA, PB connecting P with any two points on the circumference.

To prove that $PA = PB$.

Proof. 1. $\because OP \perp \odot ABC$ at C, $\therefore OP \perp AC$ and BC. Why?

2. And $\because PC \equiv PC$, and $CA = CB$, § 109

$\therefore \triangle ACP \cong \triangle BCP$, and $PA = PB$. Why?

COROLLARY. *Great-circle arcs from a pole of a circle to points on the circumference of that circle are equal.* (Why?)

451. Definitions. The length of the great-circle arc joining a pole to any point on the circumference of a circle is called the **polar distance** of the circle.

The shorter polar distance of small circles is to be understood.

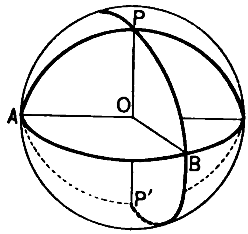
A fourth of the circumference of a great circle is called a **quadrant.**

COROLLARIES. 1. *Circles of the same sphere, having equal polar distances, are equal.* (Why?)

2. *The polar distance of a great circle is a quadrant.* (Why?)

Proposition VII.

452. Theorem. *If, on a spherical surface, each of the great-circle arcs joining a point to two other points (not the extremities of a diameter of the sphere) is a quadrant, then that point is a pole of the great circle through those points.*



Given P, A, B, three points on a spherical surface, and such that $\widehat{PA} = \widehat{PB} =$ a quadrant; A, B are not extremities of a diameter; O is the center.

To prove that P is the pole of the great circle ABO.

Proof. 1. $\because \widehat{PA} = \widehat{PB} =$ a quadrant,

$\therefore \angle POA = \angle POB =$ a rt. $\angle$. III, prop. II, cor. 2

2. $\therefore PO \perp \odot ABO$. VI, prop. VI, cor. 1

3. $\therefore P$ is a pole of $\odot ABO$. § 449, def. pole

Exercises. 690. How many points on a spherical surface determine a small circle? How many, in general, determine a great circle?

691. Prove that parallel circles of a sphere have the same poles.

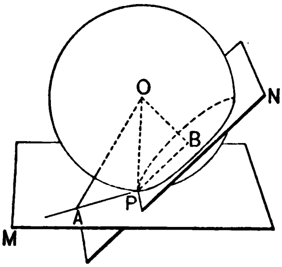
692. In the theorem: A diameter which is perpendicular to a chord bisects it, make the substitutions suggested in § 448, and prove the resulting proposition.

693. Similarly for III, prop. VI.

694. What is the locus of points at a given distance r from a fixed point C?

PROPOSITION VIII.

453. Theorem. *Of all planes through a point on a sphere the plane perpendicular to the radius drawn to that point is the only one that does not meet the sphere again.*



Given point P on a sphere with center O, and M, N, two planes respectively perpendicular and oblique to OP at P.

To prove that M does not meet the spherical surface again, but that N does.

Proof. 1. Let $OB \perp N$, and OA be any oblique to M.

Then $\because$ OP is oblique to N, Why?

$\therefore OB < OP$. VI, prop. XI

2. And $\because OP \perp M$,

$\therefore OA > OP$. VI, prop. XI

3. $\therefore$ B is within, and A without, the sphere. § 444, cor. 3

4. $\therefore$ N meets the spherical surface in more than one point. § 445, 3

5. And $\because$ A is any point in M, except P, Step 3

$\therefore$ M meets the surface only at P.

454. Definitions. A plane (or line) which, meeting a spherical surface in one point, does not meet it again, is said to be **tangent to the sphere** at that point. The point is called the **point of tangency**, or *point of contact*, and the plane (or line) is called a **tangent plane** (or **line**).

Corollaries. 1. *One and only one plane can be passed through a given point on a sphere, tangent to that sphere.* (Why?)

2. *Any tangent plane is perpendicular to the radius at the point of contact.*

For it cannot be oblique and be a tangent plane. Step 4.

3. *A plane perpendicular to a radius at its extremity on the spherical surface is tangent to the sphere.*

Exercises. 695. To find a point in a given plane, equidistant from two fixed points in that plane, and at a given distance d from a point C not in that plane. Discuss for 0, 1, 2 solutions.

696. Prove that the lateral area of any right cylinder equals the product of its altitude and the perimeter of the base. (Inscribe a prism and apply the theorem of limits.)

697. How many square feet in the surface of a cylindrical water tank, open at the top, its height being 40 ft., and its diameter 40 ft.?

698. Considering the moon as a circle of diameter 2160.6 miles whose center is 234,820 miles from the eye, what is the volume of the cone whose vertex is the eye and whose base is the full moon?

699. Find a point whose distance from a fixed point is d and whose distance from each of two intersecting planes is d'. Discuss the solution as to impossible cases, and the number of such points for possible cases.

700. Find the locus of points equidistant from two given points, and at a given distance d from a given point.

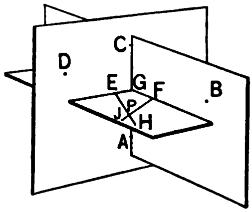
701. To determine a plane which shall pass

(1) through a given *line* and be at a given distance from a given *point*.

(2) through a given *point* and be at a given distance from a given *line*.

PROPOSITION IX.

455. Theorem. *Four points, not coplanar, determine a spherical surface.*



Given four points, A, B, C, D, not coplanar.

To prove that A, B, C, D determine a spherical surface.

Proof. 1. Draw AB, BC, CD, DA, AC.
Let E be the circumcenter of $\triangle ACD$, F of $\triangle ABC$, $EH \perp ACD$, $FJ \perp ABC$.

2. Then E, F are on the $\perp$ bisectors of AC; call these $\perp$ bisectors GE, GF. I, prop. XLI

3. And $\because EA = EC = ED$ (Why?),
$\therefore$ any point on EH is equidistant from A, C, D.
VI, prop. XI, 3

Similarly, any point on FJ is equidistant from A, B, C.

4. But $CA \perp$ plane EGF. Why?

5. $\therefore$ planes ABC, $ACD \perp EGF$. Why?

6. $\therefore$ both EH and FJ lie in plane EGF.
VI, prop. XIX, cor. 1

7. And $\because FJ$ meets EH, uniquely, as at P,
I, prop. XVII, cor. 4

$\therefore P$ is the center of a sphere whose surface passes through A, B, C, D, and there is only one such sphere.

456. Definitions. A sphere is said to be **circumscribed about a polyhedron** if the vertices of the polyhedron all lie on the spherical surface; the polyhedron is then said to be **inscribed in the sphere.**

Corollaries. 1. *Two spherical surfaces having four common points, not coplanar, coincide.*

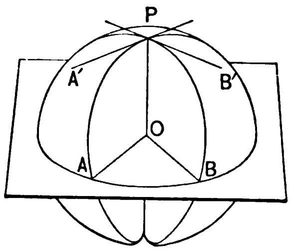
For by step 7 they have the same center, P, and the same radius.

2. *The perpendiculars to the four faces of a tetrahedron, through the circumcenters of those faces, are concurrent.*

For each of these perpendiculars passes through P, the center of the sphere whose surface is determined by the four vertices.

3. *A sphere can be circumscribed about any tetrahedron.*

457. The **angle between two great-circle arcs** is defined as being the plane angle between tangents to those arcs at their point of meeting.



E.g. the angle made by arcs AP, BP is defined as the plane angle $A'PB'$.

458. From this definition follow these *corollaries:*

1. *The angle made by two arcs has the same numerical measure as the dihedral angle of their planes.* (§ 359.)

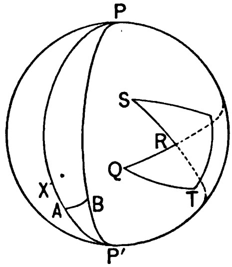
2. *An angle made by two arcs has the same numerical measure as the arc which these arcs intercept on the circumference of the great circle of which the vertex is the pole.*

That is, $\angle APB = \angle A'PB' = \angle AOB$, which has the same numerical measure as $\widehat{AB}$.

459. A **spherical polygon** is a portion of a spherical surface bounded by arcs of great circles.

The words *sides*, *angles*, *vertices*, etc., are used as with plane polygons.

460. A spherical polygon is said to be **convex** when each side produced leaves the entire polygon on the same hemisphere; otherwise it is said to be **concave.**



In the figure, ABP is a convex polygon, for if any side, as PB, is produced it leaves the entire polygon on the hemisphere to the left of PB. But $QRST$ is concave, because side SR, or QR, produced, leaves part of the polygon on one hemisphere thus formed, and part on the other.

461. COROLLARY. *No side of a convex spherical polygon is greater than a semicircumference.*

For if $AP >$ semicircumference, suppose $XP =$ a semicircumference. Then $\because$ great circles bisect each other (prop. V, cor. 5), PB must pass through X; but then PB produced would leave part of the polygon on one hemisphere and part on the other, so that it could not be convex.

462. A **lune** is a portion of a spherical surface bounded by the semicircumferences of two great circles. The **angle of a lune** is that angle toward the lune made by the bounding arcs.

In the figure, $PAP'B$ is a lune, and $\angle APB$, or $\angle BP'A$, is its angle.

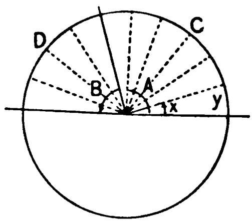
The limiting cases of a lune are evidently a semicircumference, when the angle is zero, and a spherical surface, when the angle is 360°.

463. COROLLARY. *Lunes on the same sphere, and having the same angle, are congruent.*

For one can evidently be made to coincide with the other.

Proposition X.

464. Theorem. *On the same sphere or on equal spheres lunes are proportional to their angles.*



(In this figure the eye is supposed to be looking down on the sphere from above the angle of the lune, as on the North Pole of the earth. This allows only half of each lune to be seen.)

Given two lunes, C and D, with angles A and B respectively.

To prove that $A : B = C : D$.

Proof. 1. If C and D are on different spheres, they can be placed in the relative positions shown in the figure. § 444, cor. 2

Suppose A and B divided into equal $\measuredangle$s, x, and suppose $A = nx$, and $B = n'x$.

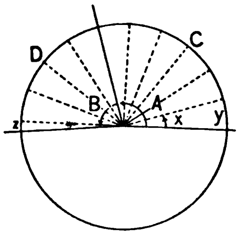
(In the figure $n = 6$, $n' = 4$.)

2. Then C is divided into n congruent lunes, y,
and D " " n' " " § 463

3. $$\therefore \frac{A}{B} = \frac{nx}{n'x} = \frac{n}{n'} = \frac{ny}{n'y} = \frac{C}{D}.$$ Why ?

Exercise. 702. The six planes perpendicular to the six edges of a tetrahedron at the mid-points of its edges, meet in a point. (Is this point the center of a particular sphere ?)

465. Proof for incommensurable case. (Compare § 410.)



1. Suppose A divided into equal $\angle$s, x, and suppose $A = nx$, while $B = n'x +$ some remainder w, such that $w < x$.

   Then C is divided into n congruent lunes, y, and D is the sum of n' congruent lunes, y, $+$ a remainder z, such that $z < y$.

2. Then B lies between $n'x$ and $(n'+1)x$, Why?
   and D " " $n'y$ " $(n'+1)y$. Why?

3. $\therefore \frac{B}{A}$ lies between $\frac{n'x}{nx}$ and $\frac{(n'+1)x}{nx}$, Why?

   while $\frac{D}{C}$ lies between $\frac{n'y}{ny}$ and $\frac{(n'+1)y}{ny}$.

4. $\therefore \frac{B}{A}$ and $\frac{D}{C}$ both lie between $\frac{n'}{n}$ and $\frac{n'+1}{n}$.

5. $\therefore \frac{B}{A}$ and $\frac{D}{C}$ differ by less than $\frac{1}{n}$. Why?

6. And $\because \frac{1}{n}$ can be made smaller than any assumed difference, by increasing n,

   $\therefore$ to assume any difference leads to an absurdity.

7. $$\therefore \frac{B}{A} = \frac{D}{C}, \text{ whence } \frac{A}{B} = \frac{C}{D}.$$

466. Definition. The solid bounded by a lune and two semi-circles is called a **spherical wedge.**

The angle of the lune is called the *angle of the wedge.*

The word *ungula* is sometimes used for spherical wedge.

Corollaries. 1. *A lune is to the spherical surface on which it lies as the angle of the lune is to a perigon.*

For the spherical surface may be considered as a lune whose angle is a perigon.

2. *A spherical wedge is to the sphere of which it is a part as the angle of the wedge is to a perigon.*

In the proof of prop. X, if we should substitute the word *wedge* for the word *lune*, and consider the sphere as a wedge whose angle is a perigon, the corollary would evidently be proved.

Exercises. 703. To draw a plane tangent to a given sphere, from a point on the sphere. (See III, prop. XXVI.) Also, to draw one from an external point.

704. To find the locus of centers of spheres whose surfaces (1) pass through two given points; (2) are tangent to two given coplanar lines; (3) are tangent to two given planes. (As special cases, the lines may be parallel and the planes may be parallel.)

705. What is the locus of the centers of spheres whose surfaces (1) pass through the vertices of a given triangle? (2) are tangent to the sides of a given triangle?

706. To find the center of a sphere whose surface includes both a given circumference and a point not in the plane of that circumference. (As a special case, suppose the point is on the perpendicular to the plane of the given circle through the center.)

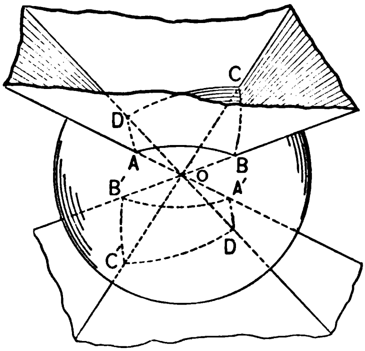
707. In the figure of prop. IX show that E, G, F, P are concyclic. Hence show that six circumferences intersect by threes in the circumcenters of the faces of a tetrahedron, and all intersect in the center of the circumscribed sphere.

708. To construct a sphere of given radius whose surface shall contain three given points.

709. Also, of given radius whose surface shall contain two given points and be tangent to a given plane.

THE RELATION OF SPHERICAL POLYGONS TO POLYHEDRAL ANGLES.

467. If the center of a sphere is at the vertex of a pyramidal space, the pyramidal surface cuts from the spherical surface two spherical polygons.



In the above figure the two polygons are *ABCD*, *A'B'C'D'*.

These polygons have their like-lettered angles and sides equal respectively.

For example, $\angle A = \angle A'$, since they have the same numerical measure as the opposite dihedral angles of planes *ADOD'A'* and *ABOB'A'*. Also, $\widehat{AB} = \widehat{A'B'}$, since the central angles *BOA* and *B'OA'* are equal.

468. But the equal elements of these polygons are arranged in reverse order. And as the polyhedral angles are called opposite and are proved (VI, prop. XXVI) symmetric, so the spherical polygons are called **opposite spherical polygons.** And since these have just been shown to have their corresponding elements equal but arranged in reverse order, they are called **symmetric spherical polygons.**

Thus all opposite polygons are symmetric; but since polygons can slide around on the sphere, it follows that symmetric polygons are not necessarily opposite, although they are congruent to opposite polygons.

469. Since the dihedral angles of the polyhedral angles have the same numerical measure as the angles of the spherical polygons, and the face angles of the former have the same numerical measure as the sides of the latter, it is evident that to each property of a polyhedral angle corresponds a reciprocal property of a spherical polygon, and *vice versa:* This relation appears by making the following substitutions:

Polyhedral Angles.	**Spherical Polygons.**
a. Vertex.	*a.* Center of Sphere.
b. Edges.	*b.* Vertices of Polygon.
c. Dihedral Angles.	*c.* Angles of Polygon.
d. Face Angles.	*d.* Sides.

470. In addition to the correspondences between polyhedral angles and spherical polygons, it will be observed that a relation exists between a straight line in a plane and a great-circle arc on a sphere. Thus, to a plane triangle corresponds a spherical triangle, to a straight line perpendicular to a straight line corresponds a great-circle arc perpendicular to a great-circle arc, etc. The word **arc** is always understood to mean **great-circle arc,** in the geometry of the sphere, unless the contrary is stated.

It is very desirable that every school have a spherical blackboard, with large wooden compasses for the drawing of both great and small circles. It is only by the use of such helps that students come to a clear knowledge of spherical geometry. If such a blackboard is at hand, it is recommended that many problems and exercises of Book I be investigated on the sphere. *E.g.* the problem, To bisect a given arc, corresponds to I, prop. XXXI, and the solutions are quite similar. Likewise the problems, To bisect a given angle, To draw a perpendicular to a given line from a given internal point, etc., have their corresponding problems in spherical geometry.

Exercise. 710. State without proof the proposition in the geometry of the sphere corresponding to the following: Every face angle of a convex polyhedral angle is less than a straight angle.

PROPOSITION XI.

471. Theorem.

(a) *In any trihedral angle, each face angle being less than a straight angle, the sum of any two face angles is greater, and their difference less, than the third angle.*

(a′) *In any spherical triangle, each side being less than a semicircumference, the sum of any two sides is greater, and their difference less, than the third side.*

Proof. In VI, prop. XXVII, with its cor. 1, (a) has been proved. Hence (a′) is also proved. § 469

PROPOSITION XII.

472. Theorem.

(a) *In any polyhedral angle, each face angle being less than a straight angle, any face angle is less than the sum of the remaining face angles.*

(a′) *In any spherical polygon, each side being less than a semicircumference, any side is less than the sum of the remaining sides.*

Proof. In VI, prop. XXVII, cor. 2, (a) has been proved. Hence (a′) is also proved. § 469

PROPOSITION XIII.

473. Theorem.

(a) *In any convex polyhedral angle the sum of the face angles is less than a perigon.*

(a′) *In any convex spherical polygon the sum of the sides is less than a circumference.*

Proof. In VI, prop. XXVIII, (a) has been proved. Hence (a′) is also proved. § 469

PROPOSITION XIV.

474. Theorem.

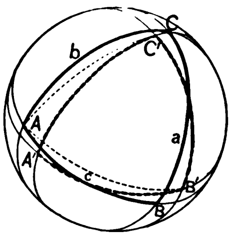
(a) *No face angle of a convex polyhedral angle is greater than a straight angle.*

(a′) *No side of a convex spherical polygon is greater than a semicircumference.*

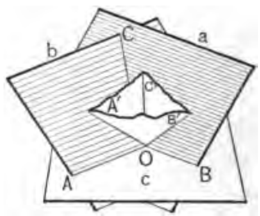
Proof. From § 461, (a′) is true.

Hence (a) is also proved. § 469

475. Definitions. If ABC is a *spherical triangle,* and A', B', C' are the *poles* of a, b, c, respectively, and if A and A', B and B', C and C' lie on the same side of a, b, c, respectively, then $\triangle A'B'C'$ is called the **polar triangle** of ABC.



If O-ABC is a *trihedral angle,* and OA', OB', OC' are *perpendiculars* to a, b, c, the faces opposite A, B, C, respectively, and if A and A', B and B', C and C' lie on the same side of a, b, c, respectively, then trihedral $\angle O$-$A'B'C'$ is called the **polar trihedral angle** of O-ABC.



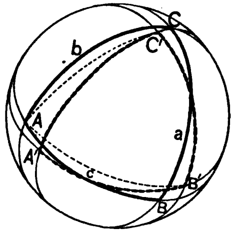
In referring to polar triangles ABC, $A'B'C'$, the above arrangement of elements will always be intended. Also, in referring to *symmetric* spherical triangles, ABC and $A'B'C'$, it will always be understood that $\angle A = \angle A'$, etc., and $\widehat{AB} = \widehat{A'B'}$, etc.

The polar triangle of ABC is often called *the polar* of ABC.

It is evident from the one-to-one correspondence of § 475, that to every proposition concerning polar triangles corresponds a proposition concerning polar trihedral angles, and *vice versa.*

PROPOSITION XV.

476. Theorem. *If one spherical triangle is the polar of a second, then the second is also the polar of the first.*



Given a spherical triangle, ABC, and $A'B'C'$ its polar.

To prove that $\triangle ABC$ is the polar of $\triangle A'B'C'$.

Proof. 1. In the figure suppose $\widehat{AC'}$, $\widehat{AB'}$, drawn.

2. Then, $\because$ B' is a pole of b,
$\therefore$ $\widehat{AB'}$ is a quadrant. Prop. VI, cor. 2
Similarly, $\because$ C' is a pole of c,
$\therefore$ $\widehat{AC'}$ is a quadrant.

3. $\therefore$ A is a pole of a'. Prop. VII
Similarly, B and C are poles of b' and c', respectively.

4. And $\because$ A, A' are on the same side of a', and so for the other vertices and sides,
$\therefore$ $\triangle ABC$ is the polar of $\triangle A'B'C'$. § 475

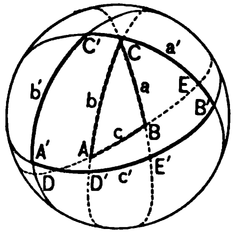
COROLLARY. *If one trihedral angle is the polar of a second, then the second is also the polar of the first.*

For from the one-to-one correspondence of § 475, the proof is evidently identical with the above.

NOTE. One triangle may fall entirely within or entirely without its polar; or one may be partly within and partly without the other. Similarly, one trihedral angle may fall entirely within or entirely without its polar trihedral angle, or may be partly within and partly without the latter.

PROPOSITION XVI.

477. Theorem. *Any angle of a spherical triangle has the same numerical measure as the supplement of the opposite side of its polar.*



Given ABC, a spherical triangle, and $A'B'C'$ its polar.

To prove that the numerical measure of any angle C is $180° - c'$; of C', $180° - c$.

Proof. 1. Suppose a, b to cut c' in E', D', respectively, and a', b' to cut c in E, D, respectively.

2. Measure of $\angle C =$ that of $\widehat{D'E'}$. § 458, 2

But $\widehat{D'E'} = \widehat{A'E'} + \widehat{D'B'} - \widehat{A'B'}$

$= 90° + 90° - \widehat{A'B'} = 180° - c'$. Why 90°?

3. Similarly for $\angle C'$, substituting A, B, D, E, for A', B', D', E', and *vice versa*, in the above proof.

COROLLARIES. 1. *If two spherical triangles are mutually equiangular, their polars are mutually equilateral; if mutually equilateral, their polars are mutually equiangular.*

2. *The sum of the* angles *of a* spherical triangle *is greater than one and less than three straight angles.*

2'. *The sum of the* dihedral angles *of a* trihedral angle *is greater than one and less than three straight angles.*

For by prop. XIII (a′), $0 < a' + b' + c' < 360°$.
∴ by subtracting from $3 \cdot 180°$,

$$3 \cdot 180° > (180° - a') + (180° - b') + (180° - c') > 180°.$$

∴ by prop. XVI, $$3 \cdot 180° > \angle A + \angle B + \angle C > 180°.$$

478. Definitions. If $ABCD$ ..... X is a *spherical polygon*, and A', B', C', D', ... are the *poles* of $\overset{\frown}{XA}$, $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CD}$, ..., respectively, and if A', B', ... lie on the same side of $\overset{\frown}{XA}$, $\overset{\frown}{AB}$, ... that the polygon does, then $A'B'C'D'$ ..... is called the **polar polygon** of $ABCD$ .....

If O-$ABCD$ ..... X is a *polyhedral* $\angle$, and OA', OB', OC', OD', ... are ⊥s to planes OXA, OAB, OBC, ..., respectively, and if A', B', ... lie on the same side of planes OXA, OAB, OBC, ... that the polyhedral angle does, then O-$A'B'C'D'$ ..... is called the **polar polyhedral angle** of O-$ABCD$ .....

***Polar trihedral angles* are also called *supplemental trihedral angles*.**

479. A spherical triangle is said to be **birectangular** if it has two right angles, **trirectangular** if it has three.

PROPOSITION XVII.

480. Theorem.

(a) *Two opposite or two symmetric trihedral angles are congruent if each has two equal dihedral angles, or two equal face angles.*

(a′) *Two opposite or two symmetric spherical triangles are congruent if each has two equal angles or two equal sides.*

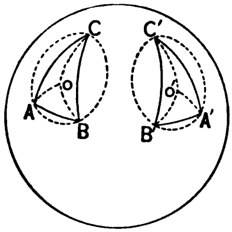
Proof for (a′). 1. Their sides and angles are respectively equal but arranged in reverse order. § 468

2. But if to the order ABC corresponds $B'A'C'$, and if $B' = A'$, then B' and A' may be interchanged.

3. Then to the order ABC will correspond $A'B'C'$, and the △s are congruent by superposition.

Proposition XVIII.

481. Theorem. *Two symmetric spherical triangles on the same sphere or on equal spheres are equal.*



Given two symmetric spherical triangles, ABC, $A'B'C'$, on the same sphere.

To prove that $\triangle ABC = \triangle A'B'C'$.

Proof. 1. The plane of A, B, C determines a small circle.

2. Let O be the pole of the $\odot$, and similarly for O' and spherical $\triangle A'B'C'$.

3. Then $\because$ side $\widehat{AB} =$ side $\widehat{A'B'}$, $\therefore$ chord $AB =$ chord $A'B'$. (In the figure they are not drawn because $\widehat{AB}$ is so nearly straight.) III, prop. III
Similarly for chords BC, $B'C'$, and CA, $C'A'$.

4. $\therefore$ plane $\triangle ABC \cong$ plane $\triangle A'B'C'$. I, prop. XII

5. $\therefore \odot ABC = \odot A'B'C'$, being circumscribed about congruent plane $\triangle$s. Why?

6. $\therefore \widehat{OA} = \widehat{OB} = \widehat{OC} = \widehat{O'A'} = \widehat{O'B'} = \widehat{O'C'}$. Why?

7. $\therefore$ spherical $\triangle$s $AOB \cong A'O'B'$, $BOC \cong B'O'C'$, $COA \cong C'O'A'$. Prop. XVII (a')

8. $\therefore \triangle ABC = \triangle A'B'C'$. Ax. 2

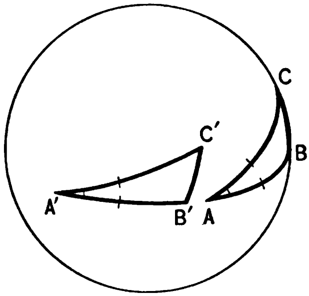
PROPOSITION XIX.

482. Theorem. *Two triangles on the same sphere, or on equal spheres, are either congruent or symmetric and equal if*

(a) *two* **sides** *and the included* **angle**

(b) *two* **angles** *and the included* **side**

of the one figure are equal to the corresponding parts of the other.



Proof. If the parts are arranged in the same order, the triangles can be brought into coincidence, as in I, props. I, II.

If they are arranged in reverse order, then one triangle is congruent to the triangle symmetric to the other. Why?

COROLLARY. *Two trihedral angles are either congruent or symmetric and equal if*

(a) *two* **face angles** *and the included* **dihedral angle**

(b) *two* **dihedral angles** *and the included* **face angle**

of the one figure are equal to the corresponding parts of the other.

For from the one-to-one correspondence of § 475, the proof is evidently identical with the above, without the labor of drawing the figures.

Proposition XX.

483. Theorem.

(a) *If a* **trihedral angle** *has two* **dihedral angles** *equal to each other, the opposite* **face angles** *are equal.*

(a′) *If a* **spherical triangle** *has two* **angles** *equal to each other, the opposite* **sides** *are equal.*

Given the $\triangle ABC$, with $\angle A = \angle B$.

To prove that $a = b$.

Proof. 1. Let $\triangle A'B'C'$ be symmetric to $\triangle ABC$, so that $a = a'$, $b = b'$, etc.

2. Then $\because \angle A = \angle B$,

$\therefore \angle A'$ must equal $\angle B'$, and the $\triangle$s are congruent and $a' = b$. Prop. XIX

3. But $\because a = a'$, and $a' = b$,

$\therefore a = b$, which proves (a′). Ax. 1

Hence (a) is also proved. § 469

Corollaries.

1. (a) *If a* **trihedral angle** *has its three dihedral angles equal, it has also its three face angles equal.*

(a′) *An equiangular* **spherical triangle** *is equilateral.*

In $\triangle ABC$, if $\angle A = \angle B$, then $a = b$; and if $\angle C$ also equals $\angle B$, c also equals b.

$\therefore$ if $\angle A = \angle B = \angle C$, $a = b = c$. Ax. 1

2. (a) *If a* **trihedral angle** *has two* **face angles** *equal to each other, the opposite* **dihedral angles** *are equal.*

(a′) *If a* **spherical triangle** *has two* **sides** *equal to each other, the opposite* **angles** *are equal.*

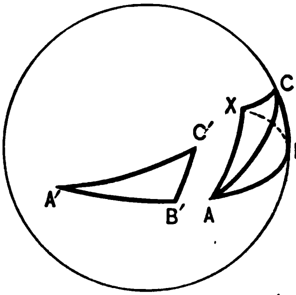
The proof is almost identical with that of I, prop. III, and hence is left for the student.

PROPOSITION XXI.

484. Theorem. *Two triangles on the same sphere, or on equal spheres, are either congruent or symmetric and equal if*

(a) *the three* **sides** (b) *the three* **angles**

of the one figure are equal to the corresponding parts of the other.



(a) **Given** $\triangle$ ABC, $A'B'C'$, mutually equilateral, the sides being arranged in the same order; also $\triangle\, ACX$ symmetric to $\triangle\, A'B'C'$.

To prove that $\triangle\, ABC \cong \triangle\, A'B'C'$, $\triangle\, ABC$ is symmetric to $\triangle\, ACX$.

Proof. 1. Place $\triangle\, ACX$ as in the figure; draw $\overset{\frown}{BX}$.

Then $\angle BXC = \angle CBX$,

and $\angle AXB = \angle XBA$. Prop. XX, cor. 2

2. $\therefore \angle AXC = \angle CBA$,

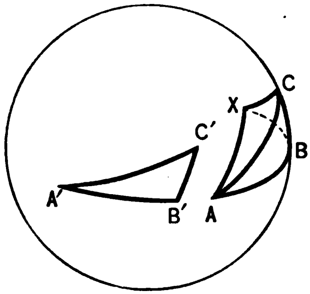
i.e. $\angle B = \angle X = \angle B'$. Ax. 3

Similarly with the other angles.

3. $\therefore \triangle\, A'B'C' \cong \triangle\, ABC$. Why?

4. And $\triangle\, ABC$ is symmetric to $\triangle\, ACX$. Why?

(b) Given (Let the student state it.)



To prove (Let the student state it.)

Proof. 1. Their polars are mutually equilateral. Why?

2. $\therefore$ their polars are congruent or symmetric. Why?

3. $\therefore \triangle ABC$ and $\triangle A'B'C'$ are mutually equilateral. Props. XV, XVI, cor. 1

4. $\therefore \triangle ABC \cong$ or symmetric to $\triangle A'B'C'$. Prop. XXI (a)

COROLLARY. *Two trihedral angles are either congruent or symmetric and equal if*

(a) *the three* **face angles** (b) *the three* **dihedral angles**

of the one figure are equal to the corresponding parts of the other.

Exercises. **711.** A plane isosceles triangle can have its equal sides of any length. Discuss as to a spherical isosceles triangle on a given sphere.

712. As with plane triangles, the pole (circumcenter) may fall outside the triangle, or on a side. Prove theorem 481 for those cases.

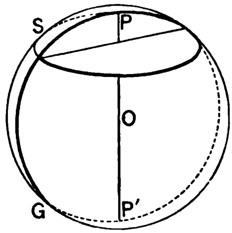
713. Prove I, prop. XII (a corresponding theorem of Plane Geometry) by the method of prop. XVIII.

714. Draw the figure of a spherical quadrilateral and its polar; also of a four-faced polyhedral angle and its polar.

715. Prove that if one spherical polygon is the polar of another, then the second is the polar of the first. State the reciprocal theorem for polyhedral angles. (The special case of the quadrilateral may be taken.)

PROPOSITION XXII.

485. Theorem. *For a great circle to be perpendicular to a small circle, it is necessary and it is sufficient that the circumference of the former pass through a pole of the latter.*



Given a small circle S, with P and P' its poles, G a great circle, and O the center of the sphere.

To prove that for G to be $\perp$ to S it is necessary and it is sufficient that its circumference pass through P.

Proof. 1. $PP' \perp S$. Def. pole

2. And if G passes through P it passes through PP', and $G \perp S$. VI, prop. XVIII

3. $\therefore$ it is *sufficient* that G contain P.

4. Furthermore it is *necessary;* for if $G \perp S$, then PP' lies in G or else $PP' \parallel G$. Why?

5. But PP' is not $\parallel$ to G, for each contains O. Why?

6. $\therefore$ it is *necessary,* and it is *sufficient,* that G contain P.

COROLLARIES. 1. *Through a point* X, *within or on a sphere, it is possible to pass one great circle perpendicular to a given circle* S, *and only one unless* X *is on the straight line through the poles of* S.

For PP' passes through the center O, and PP' and X determine a great circle $\perp S$, unless X is on PP'. (Why?) May X be without the sphere?

2. *If the circumferences of two great circles are drawn perpendicular to a third circumference, they will intersect at the poles of the circle of that third circumference.*

486. Definition. If a great circle is perpendicular to a small circle, their **circumferences** are said to be **perpendicular** to each other.

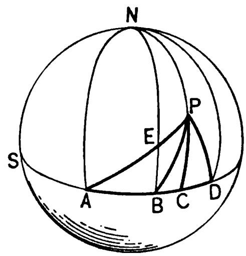
Proposition XXIII.

487. Theorem. *If from a point on a sphere arcs of great circles both perpendicular and oblique, are drawn to any circumference, then,*

1. The shorter perpendicular is less than any oblique;

2. Two obliques cutting off equal arcs from the foot of this perpendicular are equal;

3. Of two obliques cutting off unequal arcs from the foot of this perpendicular, the one cutting off the greater arc is the greater.



Given S, any circle of a sphere; P any point on the spherical surface; minor arcs, $\overset{\frown}{PC} \perp$ circumference S, $\overset{\frown}{PA}$, $\overset{\frown}{PB}$, $\overset{\frown}{PD}$ obliques; $\overset{\frown}{BC} = \overset{\frown}{CD}$, and $\overset{\frown}{AC} > \overset{\frown}{CD}$ or its equal $\overset{\frown}{BC}$.

To prove that (1) $\overset{\frown}{PC} < \overset{\frown}{PB}$;

(2) $\overset{\frown}{PB} = \overset{\frown}{PD}$;

(3) $\overset{\frown}{PA} > \overset{\frown}{PD}$ or its equal $\overset{\frown}{PB}$.

Proof. 1. Suppose N the pole of S, on the same side of S as P; draw $\widehat{NA}$, $\widehat{NB}$, $\widehat{ND}$.

2. $\widehat{CP}$ produced passes through N. Prop. XXII, cor. 2

3. $\widehat{NB}$, or its equal $\widehat{NC}$, $< \widehat{NP} + \widehat{PB}$. Prop. XI, (a′)

4. $\therefore \widehat{PC} < \widehat{PB}$. Why?

5. The radius of $\odot S$ through C is $\perp$ to chord BD and bisects it as at Q (not shown in figure). Why?

6. $\therefore DB \perp$ plane NPC. Why?

7. $\therefore$ plane $\angle s\ DQP = PQB$, Why?

and plane $\triangle s\ QPD = QPB$. I, prop. I

8. $\therefore \widehat{PB} = \widehat{PD}$. III, prop. IV

9. $\widehat{PE} + \widehat{EB} > \widehat{PB}$, and $\widehat{EA} > \widehat{EB}$. Why?

10. $\therefore \widehat{PE} + \widehat{EA}$, or $\widehat{PA}$, $> \widehat{PB}$.

488. Definition. The excess of the sum of the angles of a spherical n-gon over $(n-2)$ straight angles is called the **spherical excess** of the n-gon.

Hence the spherical excess of a 2-gon (lune), 3-gon (triangle), 4-gon, ... is the excess of the sum of its angles over 0, 1, 2, ... straight angles.

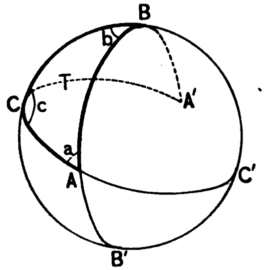
Exercises. 716. Prove that if in prop. XVI the word *polygon* is substituted for *triangle*, the resulting theorem is true, and state the corollary that follows from it, analogous to corollary 1 of prop. XVI.

717. What is meant by the spherical excess of a spherical decagon? What is the spherical excess, in degrees, of a triangle whose angles are 75°, 90°, 100°?

718. What is the spherical excess, in radians, of a triangle whose angles are 80°, 90°, 100°? Also of a triangle whose angles are 1, 2, and 3 radians, respectively?

Proposition XXIV.

489. Theorem. *A spherical triangle equals a lune whose angle is half the spherical excess of the triangle.*



Given T, a spherical triangle, with angles a, b, c.

To prove that $T =$ a lune whose angle is $\frac{1}{2}(a + b + c - \text{st.} \angle)$.

Proof. 1. Let A, B, $C =$ lunes of $\angle$s a, b, c, respectively (in the figure they are AA', BB', CC'), and $S =$ surface of sphere.

2. $\triangle$s $AB'C'$ and $A'BC$ are mutually equilateral, for $\widehat{AC'} + \widehat{CA} =$ semicircumference $= \widehat{A'C} + \widehat{CA}$; hence $\widehat{A'C} = \widehat{AC'}$, and so for the other sides. Ax. 3

3. $\therefore \triangle AB'C' = \triangle A'BC$, so that $T + \triangle AB'C' =$ lune A. Prop. XXI

4. $\therefore A + (B - T) + (C - T) = \frac{1}{2} S$, Ax. 8

or $T = \frac{1}{2}(A + B + C - \frac{1}{2} S)$. Axs. 3, 7

5. But $\because \frac{1}{2} S =$ a lune whose $\angle$ is a st. $\angle$, § 462

$\therefore T =$ a lune whose $\angle$ is $\frac{1}{2}(a + b + c - \text{st.} \angle)$.

Corollary. *A spherical polygon equals a lune whose angle is half the spherical excess of the polygon.*

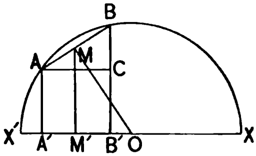
For the polygon can be cut into triangles as in Plane Geometry.

The practical method of measuring a spherical polygon is given in § 493, cor. 3.

4. THE MENSURATION OF THE SPHERE.

PROPOSITION XXV.

490. Theorem. *The area of the surface of a sphere of radius* r *is* $4\pi r^2$.



Proof. 1. A semicircle cut off by a diameter $X'X$, revolving about $X'X$ as an axis, generates a sphere.

2. Let AB be one of a number of chords inscribed in arc XBX', forming half of a regular polygon.
   Let $OM \perp AB$, thus bisecting it; III, prop. V
   let AA', MM', BB', all be $\perp$ to $X'X$, and $AC \parallel X'X$.

3. Then AB, revolving about axis $X'X$, generates the surface $l = 2\pi \cdot AB \cdot M'M$. Prop. III, cor. 2

4. But $\because \triangle ACB \backsim \triangle MM'O$, Why?
   $\therefore OM : M'M = AB : AC = AB : A'B'$.

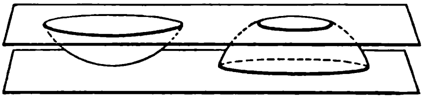
5. $\therefore AB \cdot M'M = A'B' \cdot OM$. IV, prop. I

6. $\therefore l = 2\pi \cdot A'B' \cdot OM$. Subst. in 3

7. Summing for all frustums, including two cones, the sum of the lateral surfaces $= 2\pi \cdot OM \cdot (X'A' + A'B' + \cdots\cdots) = 2\pi \cdot OM \cdot 2r$. Axs. 2, 8

8. But if the number of sides increases indefinitely, the sum of the lateral surfaces $\doteq$ surface of sphere, s, and $OM \doteq r$;
   $\therefore s = 2\pi \cdot r \cdot 2r = 4\pi r^2$. IV, prop. IX, cor. 1

491. Definitions. That part of a spherical surface which is included between two parallel planes which cut or touch the surface, is called a **zone.**



Zones and spherical segments. In first figure, lower base is zero.

The solid bounded by the zone and the two parallel planes is called a **spherical segment.**

The distance between the two parallel planes determining a zone and a spherical segment is called the **altitude** of the zone and the segment.

The circumferences in which the planes intersect the spherical surface are called the **bases of the zone,** and the circles are called the **bases of the segment.**

In case of tangent planes the bases may one or both reduce to zero. If one base only reduces to zero, the zone, or segment, is said to have *one base.*

492. Definition. As a plane angle is often said to be measured by the ratio of the intercepted arc to the whole circumference (§ 256), so **a polyhedral angle is said to be measured** by the ratio of the intercepted spherical polygon to the whole spherical surface.

The practical method of measuring a polyhedral angle is given in § 493, cor. 4.

493. COROLLARIES. 1. *The area of a zone of altitude* a, *on a sphere of radius* r, *is* $2\pi ra$.

For, prop. XXV, step 7, the sum of the lateral surfaces may approach as their limit a zone, in which case $X'A' + A'B' + \cdots \doteq a$, and $OM \doteq r$.

2. *The area of a lune of angle* α *(expressed in radian measure) on a sphere of radius* r, *is* $2\alpha r^2$.

By prop. X, $l : 4\pi r^2 = \alpha : 2\pi$.

3. *The area of a spherical polygon of spherical excess* α *(expressed in radian measure) is* αr^2.

For by prop. XXIV, the polygon equals the lune whose angle is $\alpha/2$.
$\therefore$ the area $= 2 \cdot \frac{\alpha}{2} \cdot r^2 = \alpha r^2$, by cor. 2.

4. *The measure of a polyhedral angle whose intercepted spherical polygon has a spherical excess* α *is* $\frac{\alpha}{4\pi}$.

For by definition, § 490, it is $\frac{\alpha r^2}{4\pi r^2}$.

5. *The area generated by a chord of a circle revolving about a diameter which does not cut it, equals* 2π *times the product of its projection on that diameter, and the distance from the center to the chord.* (Why?)

6. *The areas of two spheres are proportional to the squares of their radii.*

For $\frac{a}{a'} = \frac{4\pi r^2}{4\pi r'^2} = \frac{r^2}{r'^2}$.

Exercises. 719. What is the area of a spherical triangle the sum of whose angles is 4 radians, on a sphere of radius 1 ft.?

720. Also of one the sum of whose angles is 270°, r being 2 ft.?

721. Also of one the sum of whose angles is 180°, on any sphere?

722. Also of one the sum of whose angles is 237° 29′, r being 10 in.?

723. Also of a spherical quadrilateral the sum of whose angles is 417° 29′, on a sphere of radius 2 in.?

724. Also of a spherical pentagon the sum of whose angles is 4 straight angles, on a sphere of radius 5 in.?

725. What is the measure in radians of a polyhedral angle the spherical excess of whose intercepted spherical polygon is 8π?

726. What is the ratio of a trihedral angle the sum of the angles of whose intercepted spherical triangle is 1.5π radians, to a tetrahedral angle the sum of the angles of whose intercepted spherical quadrilateral is 2.5π radians?

727. What is the area of a spherical triangle whose angles are 70°, 80°, 90°, on a sphere of diameter 20 in.?

728. Show that a trirectangular triangle is its own polar.

729. The locus of points on a sphere, from which great-circle arcs perpendicular to the arms of an angle are equal, is the great-circle arc bisecting that angle.

PROPOSITION XXVI.

494. Theorem. *Two solids lying between two parallel planes, and such that the two sections made by any plane parallel to the given planes are equal, are themselves equal.*

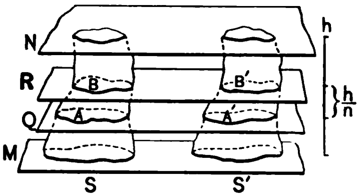


FIG. 1.

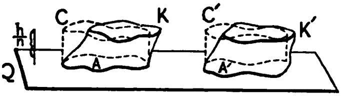


FIG. 2.

Given two solids, S, S', lying between parallel planes, M, N, and such that the two sections A, A', or B, B', ..., made by any plane Q, or R, ..., are equal, *i.e.* $A = A'$, $B = B'$, ...

To prove that $S = S'$.

Proof. 1. Let K, K' be two segments of S, S', lying between the sections A and B, and A' and B'; let the altitude of K, K' be $1/n$ of the altitude h of S and S'.

2. Suppose two straight lines to move so as always to be perpendicular to Q, and to touch the perimeters of A, A', thus generating two cylinders (or prisms, or combinations of cylinders and prisms) of altitude h/n as in Fig. 2. As the volumes of both prisms and cylinders are expressed by the same formula, $v = bh$, we may speak of these solids as cylinders, C, C'.

3. Then $C = C'$, since they have equal bases and altitudes; and so for other pairs of cylinders described in the same way, with altitude h/n.

4. $\therefore$ the sum of the solids like $C =$ the sum of the solids like C', whatever n equals.

5. But if n increases indefinitely, h/n decreases indefinitely, and it is evident that the sum of the solids like $C \doteq S$, while the sum of the solids like $C' \doteq S'$.

6. $\therefore S = S'$. IV, prop. IX, cor. 1

495. This important proposition is known as **Cavalieri's theorem.** It will be seen that VII, prop. XV, is merely a special case of this proposition. We shall base the mensuration of the volume of the sphere upon it. Solids of this kind are often called **Cavalieri bodies.**

Exercises. 730. A spherical triangle is to the surface of the sphere as the spherical excess is to eight right angles.

731. The locus of points on a sphere, from which great-circle arcs to two fixed points on the sphere are equal, is the circumference of a great circle perpendicular to the arc joining those points at its mid-point.

732. There is evidently a proposition of plane geometry analogous to Cavalieri's theorem, beginning, "Two plane surfaces lying between two parallel lines ....." State this proposition and prove it.

733. From ex. 732 prove that triangles having equal bases and equal altitudes are equal.

734. What is the ratio of the surface of a sphere to the *entire* surface of its hemisphere?

735. Prove that the areas of zones on equal spheres are proportional to their altitudes.

736. Find the ratio of the surfaces of two spheres, in terms of their radii, r_1 and r_2.

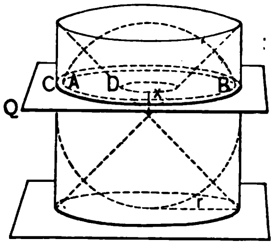
737. What is the ratio of the area of a great circle of a sphere to the area of its spherical surface?

738. If a meter is 0.0000001 of a quadrant of the earth's circumference, and the earth is assumed to be a sphere, how many square myriameters of surface has the earth?

739. What is the radius of the sphere whose area is 1 square unit? Answer to 0.001.

Proposition XXVII.

496. Theorem. *The volume of a sphere of radius* r *is expressed by the formula* $v = \frac{4}{3}\pi r^3$.



Proof. 1. Suppose the sphere circumscribed by a cylinder, and suppose two cones formed with the bases of the cylinder as their bases, and their vertices at the center of the sphere.

Suppose the solid to be cut by a plane Q, parallel to the bases, and x distant from the center of the sphere.

2. Then since x also equals the radius of the ⊙ cut from the cone, because the altitude of the cone equals the radius of its base,

$\therefore$ area of ring CD between cone and cylinder

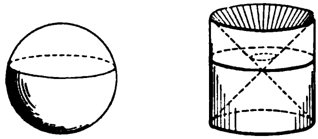
$$= \pi r^2 - \pi x^2,$$
$$= \pi (r^2 - x^2).$$

3. But the area of the ⊙ AB cut from the sphere is also $\pi (r^2 - x^2)$, because its radius is $\sqrt{r^2 - x^2}$.

4. $\therefore$ the sphere and the difference between the cone and cylinder are two Cavalieri bodies, and $\therefore$ they are equal. Prop. XXVI

5. $\therefore v = \pi r^2 \cdot 2r - \pi r^2 \cdot \frac{2r}{3} = \frac{4}{3}\pi r^3.$ Why?

COROLLARIES. 1. *The volume of a sphere equals two-thirds the volume of the circumscribed cylinder.* (Archimedes's theorem.)



The area of the circumscribed cylinder is evidently $\pi r^2 \cdot 2r$, or $2\pi r^3$. And $\frac{4}{3}\pi r^3$ is $\frac{2}{3}$ of $2\pi r^3$.

2. *The volume of a sphere equals the product of its surface by one-third of its radius.*

For the surface is $4\pi r^2$, prop. XXV; and $\frac{4}{3}\pi r^3 = \frac{1}{3}r \cdot 4\pi r^2$.

3. *The volumes of two spheres are proportional to the cubes of their radii.*

4. *The volume of a spherical segment of one base, of altitude* a, *is expressed by the formula* $v = \frac{1}{3}\pi a^2(3r - a)$.

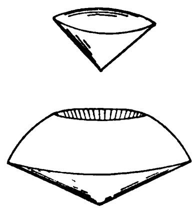
For, as in the theorem, it equals the difference between a circular cylinder of radius r and altitude a, and the frustum of a cone, of the same altitude and with bases of radii r and $(r - a)$.

$$\therefore v = \pi r^2 a - \tfrac{1}{3}\pi a[r^2 + (r-a)^2 + r(r-a)] \qquad \text{Prop. IV, cor. 1}$$
$$= \tfrac{1}{3}\pi a^2(3r - a).$$

497. Definitions. A **spherical sector** is the portion of a sphere generated by the revolution of a circular sector about any diameter of its circle as an axis.

The **base** of the spherical sector is the zone generated by the arc of the circular sector, and the **altitude** is the altitude of that zone.

If the base of the spherical sector is a zone of one base only, the spherical sector is called a **spherical cone**.



Spherical sectors. The upper one a spherical cone.

COROLLARIES. 1. *The volume of a spherical cone, whose base* b *has an altitude* a, *is expressed by the formula* $v = \frac{2}{3}\pi r^2 a$, *or* $v = \frac{1}{3} br$.

For it evidently equals the sum of a cone and a spherical segment of one base. What does the latter equal, by § 496, cor. 4? Show that the cone $= \frac{1}{3}\pi(r-a)[r^2-(r-a)^2]$. Then add the results, and show that the sum is $\frac{2}{3}\pi r^2 a$. But $b = 2\pi ra$. (Why?)

2. *The volume of a spherical sector, whose base is* b *and altitude* a, *is expressed by the formula* $v = \frac{2}{3}\pi r^2 a$, *or* $v = \frac{1}{3} br$.

For it equals the difference between two spherical cones.
Suppose these to have altitudes a_1, a_2,
and bases b_1, b_2,
and volumes v_1, v_2, respectively.
Then $v = v_1 - v_2 = \frac{2}{3}\pi r^2 a_1 - \frac{2}{3}\pi r^2 a_2 = \frac{2}{3}\pi r^2(a_1 - a_2)$.
But $a_1 - a_2 = a$. (Why?)
$\therefore v = \frac{2}{3}\pi r^2 a$. Now show that $v = \frac{1}{3} br$.

Exercises. **740.** Show that if the directrix of a cylinder is the circumference of a great circle of a sphere, and the generatrix is perpendicular to that circle, and the bases of the cylinder are circles tangent to the sphere, then the cylinder may be said to be circumscribed about the sphere.

741. After considering ex. 740, show that the surface of a sphere is two-thirds the entire surface of the circumscribed cylinder. (Archimedes.)

742. Find the ratio of a spherical surface to the cylindrical surface of the circumscribed cylinder.

743. What is the radius of that sphere the number of square units of whose area equals the number of linear units in the circumference of one of its great circles?

744. What is the ratio of the entire surface of a cylinder circumscribed about a sphere to the entire surface of its hemisphere?

745. What is the area of the entire surface of a spherical segment the radii of whose bases are r_1, r_2, the radius of the sphere being r?

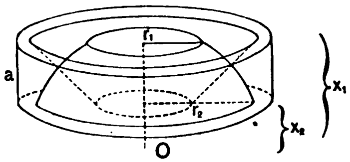
746. A cone has for its base a great circle of a sphere, and for its vertex a pole of that circle. Find the ratio of the curved surfaces of the cone and hemisphere; of the entire surfaces.

747. Show that the area of a zone of one base (the other base is zero) equals that of a circle whose radius is the chord of the generating arc.

PROPOSITION XXVIII.

498. Theorem. *The volume of a spherical segment of altitude* a, *whose bases have radii* r_1, r_2, *is expressed by the formula*

$$v = \tfrac{1}{6}\pi a[3(r_1^2 + r_2^2) + a^2], \text{ or}$$
$$v = \tfrac{1}{2}\pi a(r_1^2 + r_2^2) + \tfrac{1}{6}\pi a^3.$$



Proof. 1. Let the above figure represent a segment cut from the figure of prop. XXVII.

2. Then if the distances of the circles of radii r_1 and r_2, from the center O, are x_1 and x_2, respectively, the radii of the bases of the frustum of the cone are x_1 and x_2. Why?

3. $v =$ cylinder $-$ frustum, Prop. XXVII, step 4

$= \pi r^2 a - \frac{1}{3}\pi a(x_1^2 + x_2^2 + x_1x_2)$ Prop. IV, cors. 1, 6

$= \frac{1}{6}\pi a(6r^2 - 2x_1^2 - 2x_1x_2 - 2x_2^2)$

$= \frac{1}{6}\pi a[3(r^2 - x_1^2) + 3(r^2 - x_2^2) + (x_1 - x_2)^2]$.

4. But $\because a = x_1 - x_2$, and $r_1^2 = r^2 - x_1^2$,

and $r_2^2 = r^2 - x_2^2$,

$\therefore v = \frac{1}{6}\pi a[3(r_1^2 + r_2^2) + a^2]$

$= \frac{1}{2}\pi a(r_1^2 + r_2^2) + \frac{1}{6}\pi a^3$.

Exercise. 748. Within an equilateral triangle of side s is inscribed a circle; the triangle revolves about one of its axes of symmetry, thus generating a sphere and a cone. Find the ratio of their curved surfaces.

749. Also find the ratio of their entire surfaces.

5. SIMILAR SOLIDS.

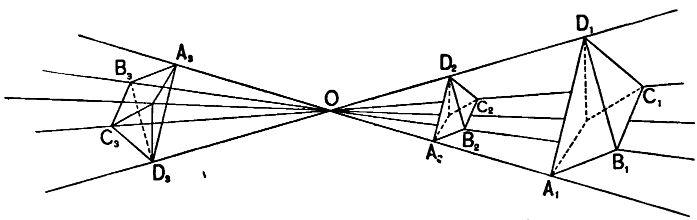
499. The definitions of **similar systems of points** and **similar figures** given in §§ 257, 258 are not limited to plane figures. The lines forming the pencil need not be coplanar. In case they are not coplanar, a pencil of lines is often called a **sheaf of lines.** The similar figures may then be plane, or they may be curved surfaces, or solids, etc. The definitions and corollaries on pages 182–184 are therefore the same for solid figures as for plane, and should be reviewed as part of this section.

Polyhedra which have equal face and equal dihedral angles, and equal edges, but have these parts arranged in reverse order, are said to be **symmetric.**

The polyhedral angles are then respectively symmetric.

PROPOSITION XXIX.

500. Theorem. *If two polyhedra are similar, their corresponding face and dihedral angles are equal, their corresponding polyhedral angles are either congruent or symmetric, and their corresponding edges are in proportion, the constant ratio being the ratio of similitude.*



Given two similar polyhedra, $A_1B_1C_1$ ..... and $A_2B_2C_2$ ....., or $A_1B_1C_1$ ..... and $A_3B_3C_3$ .....

To prove that (1) $\angle B_1A_1D_1 = \angle B_2A_2D_2$ or $\angle B_3A_3D_3$;

(2) dihedral angle with edge $A_1B_1 =$ dihedral angle with edge A_2B_2, or with edge A_3B_3;

(3) polyhedral angle $A_1 \cong$ polyhedral angle A_2 and is symmetric to polyhedral angle A_3 as arranged in the figure; and

(4) $A_1B_1 : A_2B_2 =$ the ratio of similitude.

Proof. 1. Let the polyhedra be placed in perspective (§ 259), O the center of similitude, $A_1B_1C_1$ ..... and $A_2B_2C_2$ ..... on one side of O, and $A_3B_3C_3$ ..... on the other.

2. Then as in IV, prop. XX, $A_1B_1 \parallel A_2B_2 \parallel A_3B_3$ and $D_1A_1 \parallel D_2A_2 \parallel D_3A_3$.

3. $\therefore \angle B_1A_1D_1 = \angle B_2A_2D_2 = \angle B_3A_3D_3$, and similarly for other face angles, which proves (1). VI, prop. V

4. The trihedral $\angle A_1 \cong \angle A_2$ because the face $\angle$s are respectively equal and similarly placed, and is symmetric to $\angle A_3$ because the face angles are respectively equal and placed in reverse order.
Prop. XXI, cor.

5. So for the other trihedral $\angle$s. And $\because$ polyhedral $\angle$s, as D_1, D_2, D_3, can be cut into congruent or symmetric trihedral $\angle$s similarly placed, as by the planes $A_1C_1D_1$, $A_2C_2D_2$, $A_3C_3D_3$, they too are congruent or symmetric.

6. $\therefore$ the dihedral $\angle$s are equal, which proves (2), and the corresponding polyhedral $\angle$s are congruent or symmetric, which proves (3).

7. The corresponding edges, as A_1B_1, A_2B_2, A_3B_3, being corresponding sides of similar $\triangle$s OA_1B_1, OA_2B_2, OA_3B_3, have the ratio of similitude, which proves (4).

COROLLARIES. 1. *If the ratio of similitude is 1, the polyhedra are either congruent or symmetric.*

2. *Corresponding faces of similar polyhedra are proportional to the squares of any two corresponding edges.*

Step 7, and V, prop. IV.

PROPOSITION XXX.

501. Theorem. *Two similar polyhedra can be divided into the same number of tetrahedra similar each to each and similarly placed.*

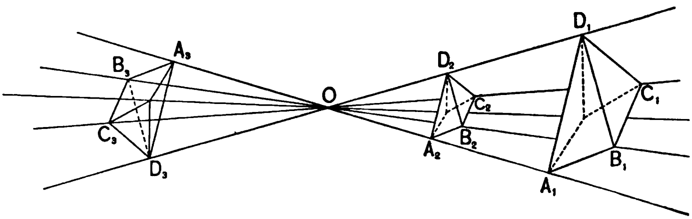
Proof. 1. In the figure below, the plane of A_1, C_1, D_1 and the plane of A_2, C_2, D_2 cut off tetrahedra $A_1B_1C_1D_1$, $A_2B_2C_2D_2$.

2. Any point P_1 in the one has a corresponding point P_2 in the other, such that $OP_1 : OP_2 =$ the ratio of similitude. Why?

3. Hence the tetrahedra are similar.

PROPOSITION XXXI.

502. Theorem. *The volumes of similar polyhedra are to each other as the cubes of their corresponding edges.*



Given the similar polyhedra $A_1B_1C_1$ ....., $A_2B_2C_2$ ....., $A_3B_3C_3$ ....., having volumes v_1, v_2, v_3, respectively.

To prove that $v_1 : v_2 = A_1B_1^3 : A_2B_2^3$, $v_1 : v_3 = A_1B_1^3 : A_3B_3^3$.

Proof. 1. Place the polyhedra in perspective, as in the figure, the center of similitude being O.

Divide the polyhedra into similar tetrahedra, similarly placed, $A_1B_1C_1D_1$, $A_2B_2C_2D_2$, $A_3B_3C_3D_3$, being corresponding tetrahedra. Prop. XXX

Let t_1, t_2, t_3 represent the volumes of these respective tetrahedra, p_1, p_2, p_3 their altitudes from D_1, D_2, D_3, and a_1, a_2, a_3 the areas of $\triangle\triangle$ $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$.

2. Then $\because t_1 = \frac{1}{3} p_1a_1$,

and $t_2 = \frac{1}{3} p_2a_2$,

$\therefore t_1 : t_2 = p_1a_1 : p_2a_2$.

3. But $a_1 : a_2 = A_1B_1^2 : A_2B_2^2$, V, prop. IV

and $p_1 : p_2 = D_1A_1 : D_2A_2 = A_1B_1 : A_2B_2$. IV, prop. XX

4. $\therefore p_1a_1 : p_2a_2 = A_1B_1^3 : A_2B_2^3$. IV, prop. VII, cor.

5. $\therefore t_1 : t_2 = A_1B_1^3 : A_2B_2^3$. From steps 2, 4

Similarly the other tetrahedra are proportional to the cubes of their corresponding edges, which edges are proportional to the particular edges A_1B_1 and A_2B_2.

6. $\therefore$ the sum of the tetrahedra making up the polyhedron $A_1B_1C_1$..... has the same ratio to the sum of the tetrahedra making up the polyhedron $A_2B_2C_2$..... as $A_1B_1^3$ has to $A_2B_2^3$, or

$$v_1 : v_2 = A_1B_1^3 : A_2B_2^3.$$

Similarly
$$v_1 : v_3 = A_1B_1^3 : A_3B_3^3,$$
$$v_2 : v_3 = A_2B_2^3 : A_3B_3^3,$$
$$= B_2C_2^3 : B_3C_3^3,$$
$$= C_2D_2^3 : C_3D_3^3, \text{ etc.}$$

Proposition XXXII.

503. Theorem. *Any two spheres are similar.*

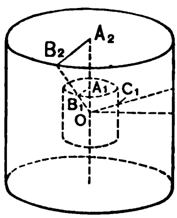
Proof. Let the spheres be placed in concentric position.

Then ∵ the ratio of their radii is constant any point on the surface of the one has on the surface of the other its similar point, with respect to the center, the ratio being $r_1 : r_2$.

∴ the spheres are similar.

Proposition XXXIII.

504. Theorem. *Two right circular cylinders are similar if their elements have the same ratio as the radii of their bases.*



Proof. 1. Let the cylinders have the radii r_1, r_2, and the altitudes h_1, h_2, respectively, and be placed with their axes in the same line, their mid-points coinciding at O.

Let the semi-altitudes be OA_1, OA_2, and let a line from O cut the bases in B_1, B_2, not necessarily on the circumferences, and one from O cut the cylindrical surfaces in C_1, C_2, respectively.

2. Then ∵ the altitudes are proportional to the radii,

$$\therefore OA_1 : OA_2 = r_1 : r_2.$$

And $\because A_1B_1 \parallel A_2B_2$, Why?

$\therefore OB_1 : OB_2 = OA_1 : OA_2 = r_1 : r_2.$ IV, prop. X

3. And $\because$ the axes coincide $\therefore$ the elements are parallel, and $\therefore OC_1 : OC_2 = r_1 : r_2.$

$\therefore$ the points of the respective cylinders are similar with respect to O as a center.

COROLLARIES. 1. *The areas of the cylindrical surfaces of two similar cylinders are proportional to the squares of their altitudes.*

For $a_1 = 2\pi r_1 h_1$ and $a_2 = 2\pi r_2 h_2$.

$$\therefore \frac{a_1}{a_2} = \frac{r_1 h_1}{r_2 h_2}.$$

But $\because \dfrac{r_1}{r_2} = \dfrac{h_1}{h_2}$ by prop. XXXIII,

$$\therefore \frac{a_1}{a_2} = \frac{h_1^2}{h_2^2}.$$

2. *The volumes of two similar right circular cylinders are proportional to the cubes of their altitudes.*

PROPOSITION XXXIV.

505. Theorem. *Two right circular cones are similar if their altitudes have the same ratio as the radii of their bases.*

Place the bases in concentric position. The proof is then so similar to that of prop. XXXIII that it is left for the student.

COROLLARIES. 1. *The areas of the surfaces of two similar right circular cones are proportional to the squares of their altitudes.*

2. *The volumes of two similar right circular cones are proportional to the cubes of their altitudes.*

EXERCISES.

750. The mean radii of the earth and moon are respectively 3956 miles, 1080.3 miles. Show that their volumes are as 49 to 1, nearly.

751. The mean diameter of the planet Jupiter being 86,657 miles, find the ratio of its volume to that of the earth.

752. The sun's diameter is about 109 times the earth's. Find the ratio of their volumes.

753. What is the radius of that sphere whose number of square units of surface equals the number of cubic units of volume?

754. Also of that whose number of cubic units of volume equals the number of square units of area of one of its great circles.

755. Also of that whose number of cubic units of volume equals the number of linear units of the circumference of a great circle.

756. Two planes cut a sphere of radius 1 m, at distances 0.8 m and 0.5 m from the center. Find (1) the area of the zone between them, (2) the volume of the corresponding spherical segment.

757. A solid cylinder 20 cm long and 2 cm in diameter is terminated by two hemispheres. The solid is melted and molded into a sphere. Find the diameter of the sphere.

758. A meter was originally intended to be 0.0000001 of a quadrant of the circumference of the earth. Assuming it to be such, and the earth to be a sphere, find its radius in kilometers.

759. A cone, a sphere, and a cylinder have the same altitudes and diameters. Show that their volumes are in arithmetical progression.

760. Given a sphere of radius 10. How far from its center must the eye be in order to see one-fourth of its surface?

761. If a tetrahedron is cut by a plane parallel to one of its faces, the tetrahedron cut off is similar to the first.

762. The areas of the surfaces of two similar polyhedra are proportional to the squares of their corresponding edges.

506. NUMERICAL TABLES.

FORMULÆ OF MENSURATION. The numbers refer to the pages. Abbreviations: b, base; h, altitude; r, radius; a, area; c, circumference; p, perimeter; s, slant height; v, volume; m, mid-section.

Parallelogram, 202,	$a = bh.$	Circle, 217, 224,	$c = 2\pi r.$
Triangle, 202,	$a = \frac{1}{2}bh.$		$a = \pi r^2.$
Trapezoid, 202,	$a = \frac{1}{2}(b + b')h.$	Arc, 223,	$= \alpha \cdot r.$

Parallelepiped, 307, $v = bh.$
Prism, 307, $v = bh.$
Lateral area, right prism, 298, $a = ph.$
Prismatoid, 314, $v = \frac{1}{6}h(b + b' + 4m).$
Pyramid, 313, $v = \frac{1}{3}bh.$
Lateral area, regular pyramid, 309, $a = \frac{1}{2}ps.$
Frustum of pyramid, 315, $v = \frac{1}{3}h(b + b' + \sqrt{bb'}).$
Lateral area, frustum of regular pyramid, 309, $a = \frac{1}{2}(p + p')s.$
Right circular cylinder, 324, 325, $v = bh = \pi r^2 h.$ Lateral $a = ch = 2\pi rh.$
Right circular cone, 324, 325, $v = \frac{1}{3}bh = \frac{1}{3}\pi r^2 h.$
Lateral $a = \frac{1}{2}cs = \pi rs.$
Frust. of rt. circ. cone, 325, $v = \frac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 r_2).$
Sphere, 355, 360, $v = \frac{4}{3}\pi r^3.$ $a = 4\pi r^2.$
Lune, 356, $a = 2\alpha r^2.$
Spherical polygon, 356, $a = \alpha r^2.$
Zone, 356, $a = 2\pi rh.$
Spherical segment, 363, $v = \frac{1}{6}\pi h[3(r_1^2 + r_2^2) + h^2].$
Spherical sector, 362, $v = \frac{2}{3}\pi r^2 h = \frac{1}{3}br.$

MOST IMPORTANT EXPRESSIONS INVOLVING π.

$\pi = 3.141593.$	$1/\pi = 0.31830989.$	$180°/\pi = 57°.29578.$
$\pi/4 = 0.785398.$	$\pi^2 = 9.86960440.$	$\pi/180 = 0.01745.$
$\pi/3 = 1.047198.$	$\sqrt{\pi} = 1.77245385.$	Approximate values;
$\frac{4}{3}\pi = 4.188790.$	$1/\sqrt{\pi} = 0.56418958.$	$\pi = \frac{22}{7} = 3\frac{1}{7}, \frac{355}{113}.$

CERTAIN NUMERICAL RESULTS FREQUENTLY USED.

$\sqrt{2} = 1.4142.$	$\sqrt{10} = 3.1623.$	$\sqrt[3]{5} = 1.7100.$
$\sqrt{3} = 1.7321.$	$\sqrt{\frac{1}{2}} = 0.7071.$	$\sqrt[3]{6} = 1.8171.$
$\sqrt{5} = 2.2361.$	$\sqrt[3]{2} = 1.2599.$	$\sqrt[3]{7} = 1.9129.$
$\sqrt{6} = 2.4495.$	$\sqrt[3]{3} = 1.4422.$	$\sqrt[3]{9} = 2.0801.$
$\sqrt{7} = 2.6458.$	$\sqrt[3]{4} = 1.5874.$	$\sqrt[3]{10} = 2.1544.$

507. BIOGRAPHICAL TABLE.

The following table includes only those names mentioned in this work, although numerous others might profitably be considered by the student. The history of geometry may be said to begin in Egypt, the work of Ahmes, copied from a treatise of about 2500 B.C., containing numerous geometric formulæ. The scientific study of the subject did not begin, however, until Thales visited that country, and carried the learning of the Egyptians back to Greece. The period of about four hundred years from Thales to Archimedes may be called the golden age of geometry. The contributions of the latter to the mensuration of the circle and of certain solids practically closed the scientific study of the subject in ancient times. Only a few contributors, such as Hero, Ptolemy, and Menelaus, added anything of importance during the eighteen hundred years which preceded the opening of the seventeenth century. Within the past three hundred years several important propositions and numerous improvements in method have been added, but the great body of elementary plane geometry is quite as Euclid left it. In recent times a new department has been created, known as Modern Geometry, involving an extensive study of loci, collinearity, concurrence, and other subjects beyond the present range of the student's knowledge.

The pronunciations here given are those of the Century Cyclopedia of Names. The first date indicates the year of birth, the second the year of death. All dates are A.D. unless the contrary is indicated by the sign —. The letter *c.* stands for *circa*, about, *b.* for *born*, *d.* for *died*. Numbers after the biographical note refer to pages in this work.

KEY. L. Latin, G. Greek, dim. diminutive, fem. feminine.

a *fat*,	ā *fate*,	ä *far*,	â *fall*,	ȧ *ask*,	ã *fare*,
e *met*,	ē *mete*,	ė *her*,	i *pin*,	ī *pine*,	o *not*,
ō *note*,	ö *move*,	ô *nor*,	u *tub*,	ū *mute*,	u̇ *pull*.
	ṅ French nasalizing *n*.			ċh German *ch*.	
	s̮ as in *leisure*.			t̮ as in *nature*.	

A single dot under a vowel indicates its abbreviation.

A double dot under a vowel indicates that the vowel approaches the short sound of *u*, as in *put*.

Ahmes (ä′mes). c. – 1700. Egyptian priest. Wrote the oldest extant work on mathematics 221
Anaxagoras (an-aks-ag′ō-ras). – 499, – 428. Greek philosopher and mathematician 225
Archimedes (är-ki-mē′dēz). – 287, – 212. Syracuse, Sicily. The greatest mathematician and physicist of antiquity . 87, 221, 353, 354
Aryabhatta (är-yä-bhä′ta). b. 476. One of the earliest Hindu mathematicians. Wrote on Algebra and Geometry 221

Bhaskara (bhäs′ka-ra). 12th cent. Hindu mathematician . . . 104
Brahmagupta (bräh-ma-göp′ta). b. 598. Hindu mathematician. One of the earliest Indian writers 143, 221

Carnot (kär-nō′), Lazare Nicholas Marguerite. 1753, 1823. French physicist and mathematician. Contributed to Modern Geometry 241, 242
Cavalieri (kä-vä-lē-ā′rē), Bonaventura. 1598, 1647. Prominent Italian mathematician 351
Ceulen (koi′len), Ludolph van. 1540, 1610. Dutch geometrician . 221
Ceva (chā′vä), Giovanni. 1648, c. 1737. Italian geometrician, 239, 241

Dase (dä′ze), Zacharias. 1824, 1861. Famous German computer . 221
Descartes (dā-kärt′), René. 1596, 1650. Eminent French mathematician, physicist, and philosopher. Founder of the science of Analytic Geometry 285

Euclid (ū′klid). c. – 300. Eminent writer on Geometry in the Alexandrian School, at Alexandria, Egypt. His "Elements," the first scientific text-book on the subject, is still the standard in the schools of England 76, 152, 208
Euler (oi′ler), Leonhard. 1707, 1783. Swiss. One of the greatest mathematicians of modern times 99, 108, 285, 289

Gauss (gous), Karl Friedrich. 1777, 1855. German. One of the greatest mathematicians of modern times 208, 212

Hero (hē′rō) of Alexandria. More properly Heron (hē′ron). c. – 110. Celebrated Greek surveyor and mechanician 221, 227
Hippocrates (hi-pok′ra-tēz) of Chios. b. c. – 470. Author of the first elementary text-book on Geometry 230

Jones (jōnz), William. 1675–1749. English teacher 221

Klein (klīn), Christian Felix. 1849. Professor at Göttingen . . 225

Leibnitz (līb′nits), Gottfried Wilhelm. 1646, 1716. Equally celebrated as a philosopher and a mathematician. One of the founders of the science of the Calculus 23, 182
Lindemann (lin′de-man), Ferdinand. b. 1852. German professor 225

TABLE OF ETYMOLOGIES.

THIS table includes such of the pronunciations and etymologies of the more common terms of Geometry as are of greatest value to the student. The equivalent foreign word is not always given, but rather the primitive root as being more helpful. The pronunciations and etymologies are those of the Century Dictionary. See Biographical Table, p. 372.

Abscissa (ab-sis′ạ̈). L. cut off.

Acute (a-kūt′). L. *acutus*, sharp.

Adjacent (ạ-jā′sẹnt). L. *ad*, to, + *jacere*, lie.

Angle (ang′gl). L. *angulus*, a corner, an angle ; G. *ankylos*, bent.

Antecedent (an-tē-se′dẹnt). L. *ante*, before, + *cedere*, go.

Bisect (bī-sekt′). L. *bi-*, two-, + *sectus*, cut.

Center (sen′tėr). L. *centrum*, center; G. *kentron*, from *kentein*, to prick.

Centroid (sen′troid). G. *kentron*, center, + *eidos*, form.

Chord (kôrd). G. *chorde*, string.

Circle (sir′kl). L. *circulus*, dim. of *circus* (G. *kirkos*), a ring.

Circumference (sėr-kum′fẹ-rẹns). L. *circum*, around (see Circle), + *ferre*, to bear.

Collinear (ko-lin′ẹ̄-ạ̈r). L. *com-*, together, + *linea*, line.

Commensurable (kọ-men′ṣṻ-rạ-bl). L. *com-*, together, + *mensurare*, measure.

Complement (kom′plẹ̄-mẹnt). L. *complementum*, that which fills, from *com-* (intensive) + *plere*, fill.

Concave (kon′-kāv). L. *com-* (intensive) + *cavus*, hollow.

Concentric (kọn-sen′trik). L. *con-*, together, + *centrum*, center.

Concurrent (kọn-kur′ẹnt). L. *con-*, together, + *currere*, run.

Concyclic (kon-sik′lik). L. *con-*, together, + *cyclicus*, from G. *kyklikos*, from *kyklos*, a circle, related to *kyliein*, roll (compare Cylinder).

Congruent (kong′grọ̈-ẹnt). L. *congruere*, to agree.

Consequent (kon′sẹ̄-kwent). L. *con-*, together, + *sequi*, follow.

Constant (kon′stänt). L. *con-*, together, + *stare*, stand.

Converse (kon′vėrs). L. *con-*, together, + *vertere*, turn.

Convex (kon′veks). L. *convexus*, vaulted, from *con-*, together, + *vehere*, carry.

Corollary (kor′ọ-lạ̄-ri). L. *corollarium*, a gift, money paid for a garland of flowers, from *corolla*, dim. of *corona*, a crown.

Cylinder (sil′in-dėr). G. *kylindros*, from *kyliein*, roll.

Decagon (dek′a̤-gon). G. *deka*, ten, + *gonia*, an angle.

Degree (dē-grē′). L. *de*, down, + *gradus*, step.

Diagonal (dī-ag′ọ̄-na̤l). G. *dia*, through, + *gonia*, a corner, an angle.

Diameter (di-am′e-tėr). G. *dia*, through, + *metron*, a measure.

Dihedral (dī-hē′dral). G. *di-*, two, + *hedra*, a seat.

Dimension (di-men′shọn). L. *dis-*, apart, + *metiri*, measure. See Measure.

Directrix (di-rek′triks). L. fem. of *director*, from *directus*, direct.

Dodecahedron (dō″dek-a̤-hē′drọn). G. *dodeka*, twelve, + *hedra*, a seat.

Equal (ē′kwa̤l). L. *æqualis*, equal, from *æquus*, plain.

Equiangular (ē-kwi-ang′gṳ-lär). L. *æquus*, equal, + *angulus*, angle.

Equilateral (ē-kwi-lat′ẹ-ra̤l). L. *æquus*, equal, + *latus*, side.

Equivalent (ẹ̄-kwiv′a̤-lẹnt). L. *æquus*, equal, + *valere*, be strong.

Escribed (es-krībd′). L. *e*, out, + *scribere*, write.

Excess (ek-ses′). L. *ex*, out, + *cedere*, go; *i.e.* to pass beyond.

Frustum (frus′tum). L. a piece.

Generatrix (jen′ẹ-rā-triks). L. fem. of *generator*, from *generare*, beget, from *genus*, a race.

Geometry (jẹ̄-om′e-tri). G. *ge*, the earth, + *metron*, a measure.

-gon, a termination, G. *gonia*, an angle.

Harmonic (här-mon′ik). G. *harmonia*, a concord, related to *harmos*, a joining. A line divided internally and externally in the ratio 2 : 1, is cut into segments representing 1, $\frac{2}{3}$, $\frac{1}{2}$. Pythagoras first discovered that a vibrating string stopped at half its length gave the octave of the original note, and stopped at two-thirds of its length gave the fifth. Hence the expression "harmonic division" of a line.

Hemisphere (hem′i-sfēr). G. *hemi-*, half, + *sphaira*, a sphere.

-hedron, a termination, G. *hedra*, a seat.

Hepta-, in combination, G. seven.

Hexa-, in combination, G. six.

Hexagram (hek′sa̤-gram). G. *hex*, six, + *gramma*, a line.

Hypotenuse (hī-pot′e-nūs). G. *hypo*, under, + *teinein*, stretch.

Inclination (in-kli-nā′shọn). L. *in*, on, + *clinare*, lean.

Incommensurable (in-kọ̄-men′sṳ̄-ra̤-bl). L. *in-*, not, + *com-*, together, + *mensurare*, measure.

Infinity (in-fin′i̤-ti). L. *in-*, not, + *finitus*, bounded.

Inscribed (in-skrībd′). L. *in*, in, + *scribere*, write.

Isosceles (ī-sos′e-lēz). G. *isos*, equal, + *skelos*, leg.

Lateral (lat′ẹ-ra̤l). L. *latus*, a side.

Locus (lō′kus). L. a place. Compare locality.

Lune (lūn). L. *luna*, the moon.

Major (mā′jọr). L. greater, comparative of *magnus*, great.

Maximum (mak′si-mum). L. greatest, superlative of *magnus*, great.

Mean (mēn). L. *medius*, middle.

Measure (mezh′ūr). L. *mensura*, a measuring. See Dimension.

Median (mē′di-ăn). See Mean.

Mensuration (men-sū-rā′shon). See Measure.

Minimum (min′i-mum). L. least.

Minor (mī′nor). L. less.

Nappe (nap). French, a cloth, sheet, surface.

Oblique (ob-lēk′ or ob-līk′). L. *ob*, before, + *liquis*, slanting.

Obtuse (ob-tūs′). L. *obtusus*, blunt, from *ob*, upon, + *tundere*, strike.

Octo-, octa-, in combination, L. and G., eight.

Opposite (op′ō-zit). L. *ob*, before, against, + *ponere*, put, set.

Ordinate (ôr′di-nāt). L. *ordo* (*ordin-*), a row.

Orthocenter (ôr′thō-sen-tėr). G. *ortho-*, straight, + *kentron*, center.

Orthogonal (ôr-thog′ō-nal). G. *orthos*, right, + *gonia*, an angle.

Parallel (par′a-lel). G. *para*, beside, + *allelon*, one another.

Parallelepiped (par-a-lel-e-pip′ed or -pī′ped). Gr. *parallelos*, parallel, + *epipedon*, a plane surface, from *epi*, on, + *pedon*, ground.

Parallelogram (par-a-lel′ō-gram). G. *parallelos*, parallel, + *gramma*, line.

Pedal (ped′al or pē′dal). L. *pedalis*, pertaining to the foot, from *pes* (*ped-*), foot.

Pencil (pen′sil). L. *penicillum*, a painters' pencil, a brush.

Perigon (per′i-gon). G. *peri*, around, + *gonia*, a corner, angle.

Perimeter (pē-rim′e-tėr). G. *peri*, around, + *metron*, measure.

Perpendicular (pėr-pen-dik ū-lār). L. *perpendiculum*, a plumb-line, from *per*, through, + *pendere*, hang.

Perspective (pėr-spek′tiv). L. *per*, through, + *specere*, see.

π (pī). Initial of G. *periphereia*, periphery, circumference.

Pole (pōl). G. *polos*, a pivot, hinge, axis, pole.

Polygon (pol′i-gon). G. *polys*, many, + *gonia*, corner, angle.

Polyhedron. (pol-i-hē′dron) G. *polys*, many, + *hedra*, seat.

Postulate (pos′tū-lāt). L. *postulatum*, a demand, from *poscere*, ask.

Prism (prizm). G. *prisma*, something sawed, from *priein*, saw.

Prismatoid (priz′ma-toid). G. *prisma* (*t-*), + *eidos*, form.

Projection (prō-jek′shon). L. *pro*, forth, + *jacere*, throw.

Pyramid (pir′a-mid). G. *pyramis*, a pyramid, perhaps from Egyptian *pir-em-us*, the slanting edge of a pyramid.

Quadrant (kwod′rant). L. *quadran*(*t-*)*s*, a fourth part. See Quadrilateral.

Quadrilateral (kwod-ri-lat′e-ral). L. *quattuor* (*quadri-*), four, + *latus*, (*later-*), side.

Radius (rā′di-us). L. rod, spoke of a wheel.

Ratio (rā′shiō). L. a reckoning, calculation, from *reri*, think, estimate.

Reciprocal (rē-sip′rō-kal). L. *reciprocus*, returning, from *re-*, back, and *pro*, forward, with two adjective terminations.

Rectangle (rek′tang-gl). L. *rectus*,

right, + *angulus*, an angle. See Angle.

Rectilinear (rek-ti-lin'-ä̤r). L. *rectus*, right, + *linea*, a line.

Reflex (rē'fleks or rẹ-fleks'). L. *re-*, back, + *flectere*, bend.

Regular (reg'ṳ-lä̤r). L. *regula*, a rule, from *regere*, rule, govern.

Rhombus (rom'bus). G. *rhombos*, a spinning top.

Scalene (skạ̄-lēn'). G. *skalenos*, uneven, unequal; related to *skellos*, crooked-legged.

Secant (sē'kant). L. *secare*, cut, as also Sector, Section, Segment.

Segment (seg'ment). See Secant.

Semicircle (sem'i-sėr-kl). L. *semi-*, half, + *circulus*, circle.

Similar (sim'i-lä̤r). L. *similis*, like.

Solid (sol'id). L. *solidus*, firm, compact.

Sphere (sfēr). G. *sphaira*, a ball.

Square (skwār). L. *quadra*, a square, from *quattuor*, four.

Straight (strāt). Anglo-Saxon, *streht*, from *streccan*, stretch.

Subtend (sub-tend'). L. *sub*, under, + *tendere*, stretch.

Successive (suk-ses'iv). L. *sub*, under, + *cedere*, go.

Sum (sum). L. *summa*, highest part. Compare Summit.

Superposition (sū'pėr-pọ̄-zish'ọn). L. *super*, over, + *ponere*, lay.

Supplement (sup'lẹ-mẹ̈nt). L. *sub*, under, + *plere*, fill; to fill up.

Surface (sėr-fạ̄s). L. *superficies*, surface, from *super*, above, + *facies*, form, figure, face.

Symbol (sim'bọl). G. *symbolos*, a sign by which one infers something, from *syn*, together, + *ballein*, put.

Tangent (tan'jẹ̈nt). L. *tangere*, touch.

Tetrahedron (tet-ra̤-hē'drọn). G. *tetra-*, four, + *hedra*, seat.

Theorem (thē'ō-rem). G. *theorema*, a sight, a principle contemplated.

Transversal (trȧns-vėr'sa̤l). L. *trans*, across, + *vertere*, turn.

Trapezium (trạ̄-pē'zi-um). G. *trapezion*, a table, dim. of *trapeza*, a table, from *tetra*, four, + *pous*, foot.

Trapezoid (trạ̄-pē'zoid). G. *trapeza*, table, + *eidos*, form.

Tri-, in composition, L. *tres* (*tri-*), G. *treis* (*tri-*), three. See Secant, -hedron, Angle, for meaning of trisect, trihedral, triangle.

Truncate (trung'kāt). L. *truncare*, cut off, from Old L. *troncus*, cut off, mutilated.

Unique (ụ̄-nēk'). L. *unicus*, from *unus*, one.

Vertex (vėr'teks). L. *vertere*, turn.

Zone (zōn). G. *zone*, a girdle, belt.

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